# M421 HW 6

## Due Monday Dec. 4

From Wade

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## Non-book Exercises

**1)** Let $\Gamma$ be a curve in $\mathbf{R}^3$,

$$\Gamma = \left\{ \vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \,\middle|\, t \in [0,1] \right\},$$

where $\vec{\gamma} \in \mathcal{C}^2([0,1], \mathbf{R}^3)$ satisfies $\|\vec{\gamma}'(t)\| = 1$ for all $t \in [0,1]$. Suppose that $\vec{\phi}$ and $\vec{\psi}$ are $\mathcal{C}^1([0,1], \mathbf{R}^3)$ functions which satisfy $\|\vec{\phi}(t)\| = \|\vec{\psi}(t)\| = 1$.

Find a condition on $\vec{\phi}$ and $\vec{\psi}$ such that the map $F : \mathbf{R}^3 \mapsto \mathbf{R}^3$ given by

$$F(t, s_1, s_2) = \vec{\gamma}(t) + s_1 \vec{\phi}(t) + s_2 \vec{\psi}(t),$$

defines a $\mathcal{C}^1$ coordinate system local to the curve $\gamma$. That is, find conditions which make $F$ invertible in some neighborhood of each point of $\Gamma$ with $F^{-1} \in \mathcal{C}^1$.

**2)** Let $\Gamma$ be a smooth two-dimensional submanifold of $\mathbf{R}^3$, ie.

$$\Gamma = \left\{ \vec{\gamma}(\vec{t}) = \left(\gamma_1(\vec{t}), \gamma_2(\vec{t}), \gamma_3(\vec{t})\right) \,\middle|\, \vec{t} = (t_1, t_2) \in [0,1] \times [0,1] \right\},$$

where $\vec{\gamma} \in \mathcal{C}^2([0,1] \times [0,1], \mathbf{R}^3)$, satisfies $\left\| \frac{\partial \vec{\gamma}}{\partial t_1}(\vec{t}) \times \frac{\partial \vec{\gamma}}{\partial t_2}(\vec{t}) \right\| = 1$. Let $\vec{\nu}(\vec{t})$ be a normal to $\Gamma$ at $\vec{\gamma}(t)$ choosen to be locally smooth (there are two normals at each point, choose $\nu$ consistently). Show that the map

$$F(t, s) = \vec{\gamma}(t) + s\vec{\nu}(t)$$

taking $\mathbf{R}^3$ to $\mathbf{R}^3$ is locally invertible in a neigborhood of each point $\vec{\gamma}(\vec{t}) \in \Gamma$ with a $C^1$ inverse.

P350 #8
11.4

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be diff at $(a,b)$ with $F_y(a,b) \neq 0$ and $I$ an open interval containing $a$. Prove that if $f: I \to \mathbb{R}$ is differentiable at $a$, $f(a)=b$ and $F(x, f(x))=0$ $\forall x \in I$ then

$$\frac{df}{dx}(a) = - \frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial y}(a,b)}$$

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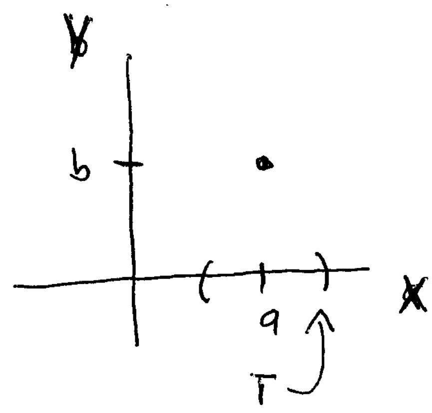


Define $G: \mathbb{R} \to \mathbb{R}^2$ by

$$G(x) = (x, f(x)) \quad \forall x \in I$$

then $G$ is diff at $a$ and

$$F(x, f(x)) = (F \circ G)(x) = 0 \quad \forall x \in I.$$

By the chain rule

$$0 = D(F \circ G)(x) = DF(G(x)) \cdot DG(x)$$

At $x = a$ we have $G(a) = (a, f(a)) = (a,b)$

$$0 = DF(G(a))\, DG(a) = \left[ \frac{\partial F}{\partial x}(a,b) \quad \frac{\partial F}{\partial y}(a,b) \right] \begin{bmatrix} 1 \\ \frac{df}{dx}(a) \end{bmatrix}$$

$$0 = \frac{\partial F}{\partial x}(a,b) + \frac{\partial F}{\partial y}(a,b) \frac{df}{dx}(a)$$

Since $\frac{\partial F}{\partial y}(a,b) \neq 0$

$$\boxed{\frac{df}{dx}(a) = - \frac{\frac{\partial F}{\partial x}(a,b)}{\frac{\partial F}{\partial y}(a,b)}}$$

p358 #7 $V^{\text{open}} \subset \mathbb{R}^n$, $f: V \to \mathbb{R}$ is $\mathcal{C}^2$

If $Df(\vec{a}) = 0$ $\exists \vec{a} \in V$, prove that if $H^{\text{compact, connected}} \subset V$, then $\exists M > 0$ ∋

$$|f(\vec{x}) - f(\vec{a})| \le M \|x - \vec{a}\|^2$$

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By Taylor Formula (11.37 p356)

$$(1) \quad f(\vec{x}) = f(\vec{a}) + \cancel{D^{(1)} f(\vec{a}, \vec{h})}^{0} + \frac{1}{2} D^{(2)} f(\vec{c}, \vec{h})$$

where $\vec{c} \in L(\vec{x}, \vec{a})$ and $\vec{h} = \vec{x} - \vec{a}$.

By assumption $D^{(1)} f(\vec{a}; \vec{h}) = Df(a)\vec{h} = 0$

by the formula on page 355

$$D^2 f(\vec{c}, \vec{h}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{c})\, h_i h_j$$

So (1) ⇒

$$f(\vec{x}) - f(\vec{a}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{c})\,(x_i - a_i)(x_j - a_j)$$

for $\vec{c} \in L(\vec{x}, \vec{a}) \subset H$. Since $H$ is compact, $\exists$ $M > 0$ ∋.

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{c}) \right| < M \quad \forall \vec{c} \in H$$

thus

$$|f(\vec{x}) - f(\vec{a})| \leq \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{c})\, (x_i - a_i)(x_j - a_j) \right|$$

$$\leq M \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - a_i|\,|x_j - a_j|$$

Young's inequality

$$\leq \frac{M}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( |x_i - a_i|^2 + |x_j - a_j|^2 \right)$$

$$\leq \frac{M}{2} \sum_{i=1}^{n} \left( n\,|x_i - a_i|^2 + \|\vec{x} - \vec{a}\|^2 \right)$$

$$|f(\vec{x}) - f(\vec{a})| \leq n M \|\vec{x} - \vec{a}\|^2$$

P368

11.6 #3 Prove $\exists$ functions $(u(x,y), v(x,y), w(x,y))$ and $r>0$ s.t.

$u,v,w$ are $\mathcal{C}^1(B_r(1,1))$, $u(1,1)=v(1,1)=-w(1,1)=1$

and satisfy

$$F(x,y,u,v,w) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{cases} u^5 + xv^2 - y + w = 0 \\ v^5 + yu^2 - x + w = 0 \\ w^4 + y^5 - x^4 \, 1 = 0 \end{cases}$$

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Since the functions $F$ are smooth and $F(1,1,1,1,-1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

we need only verify that

$$\frac{\partial(f_1\ f_2\ f_3)}{\partial(u\ v\ w)}(1,1,1,1,-1) \neq 0$$

$$\frac{\partial(f_1\ f_2\ f_3)}{\partial(u,v,w)} = \det\begin{pmatrix} 5u^4 & 2xv & 1 \\ 2uy & 5v^4 & 1 \\ 0 & 0 & 4w^3 \end{pmatrix}\Bigg|_{(1,1,1,1,-1)}$$

$$= \det\begin{pmatrix} 5 & 2 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 4 \end{pmatrix} = 4\det\begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$$

$$= 4\cdot(25-4) = 84 \neq 0.$$

Thus $\exists$ $\mathcal{C}^1$ functions $(u(x,y), v(x,y), w(x,y))$ solving

by Implicit Function Thm

$$F(x,y,u(x,y),v(x,y),w(x,y)) = 0$$

on $B_r(1,1)$ $\exists r>0$.

Nonbook #1) $\Gamma = \{ \vec{\gamma}(t) \mid t \in [0,1] \} \subset \mathbb{R}^3$

with $\vec{\gamma} \in \mathcal{C}^2([0,1], \mathbb{R}^3)$ and $\|\vec{\gamma}\,'\| = 1 \quad \forall t \in [0,1]$.

Suppose $\vec{\phi}$ and $\vec{\psi} \in \mathcal{C}^1([0,1], \mathbb{R}^3)$ satisfy $\|\vec{\phi}(t)\| = \|\vec{\psi}(t)\| = 1$.

Find conditions on $\vec{\phi}$ & $\vec{\psi}$ s.t.

$$F(t, s_1, s_2) = \vec{\gamma}(t) + s_1 \vec{\phi}(t) + s_2 \vec{\psi}(t)$$

defines a $\mathcal{C}^1$ coord system local to $\Gamma$.

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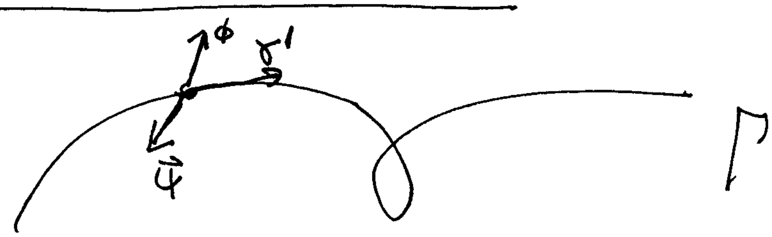


Show that for $t_0 \in [0,1]$

$\exists\, r > 0$ s.t.

$$F(t, s_1, s_2) = (x, y, z)$$

has a unique soln $(t, s_1, s_2) = F^{-1}(x, y, z)$

$$\forall \|(x, y, z) - \vec{\gamma}(t_0)\| < r$$

Since $F$ is $\mathcal{C}^1$ we need only check that

$\cancel{\Delta_F(t_0,0,0) \neq 0}$ $\cancel{\text{at } (t_0, 0, 0)}$

$$\Delta_F(t_0, 0, 0) \neq 0$$

$$\Delta_F^{(t_0,0,0)} = \det \begin{pmatrix} \gamma_1'(t_0) & \phi_1 & \psi_1 \\ \gamma_2'(t_0) & \phi_2 & \psi_2 \\ \gamma_3'(t_0) & \phi_3 & \psi_3 \end{pmatrix} \neq 0$$

this ~~implies~~ condition is equivalent to

$$\vec{\gamma}' \cdot (\vec{\phi} \times \vec{\psi}) \neq 0$$

or alternatively

$$\vec{\gamma}' \notin \operatorname{span}(\vec{\phi}, \vec{\psi})$$

If the tangent to the curve is linearly independent from $\{\vec{\Phi}(t), \psi(t)\}$ at each point $t$, then $F$ is a $\mathcal{C}^1$ coord system.

NB#2 $\quad \Gamma = \{\vec\gamma(\vec t) : \vec t = (t_1, t_2) \in [0,1]\times[0,1]\} \subset \mathbb{R}^3$

where $\vec\gamma \in \mathcal{C}^2([0,1]\times[0,1], \mathbb{R}^3)$ satisfies

$$\left\| \frac{\partial\vec\gamma}{\partial t_1} \times \frac{\partial\vec\gamma}{\partial t_2} \right\| = 1$$

Let $\vec\nu(t)$ be the local normal. Show that the map

$$F(t,s) = \vec\gamma(t) + s\,\vec\nu(t)$$

is locally invertible with a $\mathcal{C}^1$ inverse.

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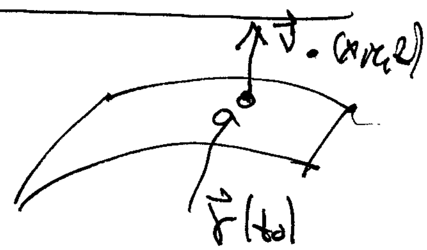
We want to solve




$$(x,y,z) = F(\vec t, s)$$

in a neighborhood of $F(\vec t_0, 0) = \vec\gamma(\vec t_0)$

$$\vec t_0 = (t_{0,1}, t_{0,2})$$

$$\boxed{F \text{ is } \mathcal{C}^1\text{, check that } \Delta_F(\vec t_0, 0) \neq 0}$$

use this fact

$$\det(A \,|\, B \,|\, C) = (A\times B)\cdot C$$

$$\det\left| \frac{\partial\gamma}{\partial t_1} \,\Big|\, \frac{\partial\gamma}{\partial t_2} \,\Big|\, \vec\nu \right| \neq 0$$

$$= \left(\frac{\partial\vec\gamma}{\partial t_1} \times \frac{\partial\vec\gamma}{\partial t_2}\right)\cdot\vec\nu$$

but $\nu = \pm\left(\frac{\partial\gamma}{\partial t_1} \times \frac{\partial\gamma}{\partial t_2}\right)$

so $\det\left| \frac{\partial\gamma}{\partial t_1}\ \frac{\partial\gamma}{\partial t_2}\ \nu \right| = \pm \left\| \frac{\partial\gamma}{\partial t_1} \times \frac{\partial\gamma}{\partial t_2} \right\|^2 = \pm 1 \neq 0$

Thus Inv. Funct thm $\Rightarrow$ F is locally invertible on $\Gamma$