M421 HW 5

Due Friday Nov. 9

From Wade

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Non-book Exercises

1) For which $\alpha > 0$ is the function

$$f(x,y) = \begin{cases} \dfrac{x^2|y|^\alpha}{x^2+|y|^3} & (x,y) \neq 0 \\ 0 & (x,y) = 0, \end{cases}$$

differentiable at zero?

2) Consider $R^\infty = \{\vec{x} = (x_1, x_2, x_3, \ldots) \big| \, x_i \in \mathbf{R}, i = 1, 2, 3, \ldots\}$. The l^2 norm on R^∞ is

$$\|\vec{x}\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

The space $l^2 = \{\vec{x} \in R^\infty \big| \, \|\vec{x}\|_2 < \infty\}$, is infinite dimensional. Show that the l^2 unit sphere, $S = \{\vec{x} \in l^2 \big| \, \|\vec{x}\|_2 = 1\}$, is closed, bounded, and **not** sequentially compact. That is, find a sequence from S which has no convergent subsequence.

Honor's Problems

3) Define the space $\mathcal{C}^1[a,b] = \{f : [a,b] \mapsto \mathbf{R} \big| \quad f \mathrm{and} f' \in \mathcal{C}[\mathrm{a,b}]\}$, and the norm

$$\|f\|_{1,1} = \int_a^b \left(|f(x)| + |f'(x)|\right) dx.$$

(a) Show there exists $M > 0$ such that for all $f \in \mathcal{C}^1[a,b]$, $\|f\|_\infty \leq M\|f\|_{1,1}$.
Hint: Use the Fundamental Theorem of Calculus.

(b) Show that if $\{f_n\}_{n=1}^\infty$ is a sequence from $\mathcal{C}^1$ and $f_n \to g$ in $\| \circ \|_{1,1}$ then $f_n \to g$ point wise.

(c) Define $W^{1,1}$ to be the set of all sequences from C^1 which are cauchy in the norm $\| \circ \|_{1,1}$. Show that for any sequence $\{f_n\}$ from C^1 which is cauchy in $\| \circ \|_{1,1}$ there is a $g \in \mathcal{C}[a,b]$ such that

$$\|f_n - g\|_1 \to 0.$$

For this reason we say that

$$W^{1,1} \subset \mathcal{C}[a,b].$$

4)(a) Let $f \in L^1[a,b]$ and $g \in W^{1,1}[a,b]$. Show that the product $fg \in L^1[a,b]$. That is, if f is represented by the $\| \circ \|_1$ cauchy sequence $\{f_n\} \subset \mathcal{C}[a,b]$ and g by the $\| \circ \|_{1,1}$ cauchy sequence $\{g_n\} \subset \mathcal{C}^1[a,b]$, then the sequence $\{h_n\}$ where $h_n = f_n g_n$ is contained in $C[a,b]$ and is cauchy in $\| \circ \|_1$.

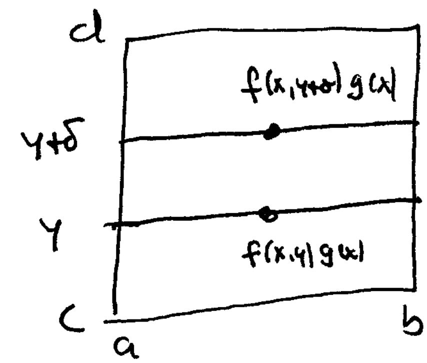
(b) In part (a), show that if g is merely in $L^1[a,b]$, then the product fg may not be in $L^1[a,b]$. That is find two sequences $\{f_n\}$ and $\{g_n\}$, both from $\mathcal{C}[a,b]$ and both cauchy in $\| \circ \|_1$ such that the "product" $\{f_n g_n\}$ is not cauchy in $\| \circ \|_1$.

HW5

Suppose that $H = [a,b] \times [c,d]$ is a rectangle $\subset \mathbb{R}^2$ and $f : H \to \mathbb{R}$ is continuous and $g : [a,b]$ is integrable.

Prove that
$$F(y) = \int_a^b g(x) f(x,y)\, dx$$
is uniformly continuous on $[c,d]$.

Proof



Since f is continuous on H and H is compact $\Rightarrow f$ is uniformly continuous on H.

So $\forall \varepsilon > 0 \ \exists \delta \ \ni$
$$\|(x,y) - (x',y')\| < \delta \Rightarrow |f(x,y) - f(x',y')| < \varepsilon$$

Since g is integrable on $[a,b] \Rightarrow g$ is bdd $\Rightarrow \exists M > 0 \ \ni$
$$\Rightarrow |g(x)| < M \quad \forall x \in [a,b].$$

Now $|y - y'| < \delta \Rightarrow$
$$|F(y) - F(y')| = \left| \int_a^b f(x,y) g(x)\, dx - \int_a^b f(x,y') g(x)\, dx \right|$$
$$= \left| \int_a^b (f(x,y) - f(x,y'))\, g(x)\, dx \right|$$
$$\leq \int_a^b \underbrace{|f(x,y) - f(x,y')|}_{\downarrow \text{ since } |y-y'|<\delta} |g(x)|\, dx$$
$$\leq \int_a^b \varepsilon \cdot M\, dx \leq \varepsilon M (b-a)$$

$\Rightarrow F$ is uniformly cont. (δ indep of y, y').

11.2 #5

Prove that if $\alpha > 1/2$ then

$$f(x,y) = \begin{cases} |xy|^\alpha \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is diff at $(0,0)$

Lets check the directional derivatives at $(0,0)$

$$D_v f(0,0) = \lim_{r\to 0^+} \frac{f(r\vec{v}) - f(0)}{r} = \lim_{r\to 0} \frac{r^{2\alpha}|v_1 v_2|^\alpha \log(r^2\|\vec{v}\|)}{r}$$

$$= \lim_{r\to 0} r^{2\alpha-1} |v_1 v_2|^\alpha \log(r^2\|\vec{v}\|)$$

$$= \begin{cases} 0 & \alpha > 1/2 \\ -\infty & \alpha \geq 1/2 \end{cases}$$

So directional derivatives do not exist for $\alpha \geq 1/2$

Check the limit for $\alpha > 1/2$ with $Df(0,0) = 0$

$$\lim_{\vec{h}\to 0} \frac{\|f(\vec{h}) - f(0) - Df(0,0)\vec{h}\|}{\|h\|} = \lim_{\vec{h}\to 0} \frac{|h_1 h_2|^\alpha \log\|h\|^2}{\|h\|}$$

$$\leq \lim_{\vec{h}\to 0} \frac{\left(\frac{1}{2}(h_1^2+h_2^2)\right)^\alpha \log\|h\|^1}{\|h\|}$$

$$\leq \left(\frac{1}{2}\right)^\alpha \lim_{h\to 0} \|h\|^{2\alpha-1} \log\|h\|^2$$

$$= 0 \quad \text{if } \alpha > 1/2$$

$\therefore$ $\boxed{Df(0,0) = 0 \quad \text{if} \quad \alpha > 1/2}$

HW5 NB #1

For which $\alpha > 0$ is

$$f(x,y) = \begin{cases} \dfrac{x^2|y|^\alpha}{x^2+|y|^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

differentiable at zero?

First consider the directional derivatives

$$D_v f(0,0) = \lim_{h\to 0^+} \frac{f(hv) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{|h|^{2+\alpha}\, v_1^2 |v_2|^\alpha}{h\left(h^2 v_1^2 + |h|^3 |v_2|^3\right)} = \lim_{h\to 0} h^{\alpha-1} \frac{v_1^2 |v_2|^\alpha}{v_1^2 + h|v_2|^3} = \begin{cases} \infty & \alpha<1 \\ |v_2|^\alpha & \alpha=1 \\ 0 & \alpha>1 \end{cases}$$

For $\alpha < 1$: $D_v f(0,0)$ is not defined $\Rightarrow$ f is not differentiable at $(0,0)$

$\alpha = 1$: $D_v f(0,0) = |v_2|^\alpha$ is not linear $\Rightarrow$ f is not differentiable at $(0,0)$

$D_{-v} f(0,0) \neq -D_v f(0,0)$

For $\alpha > 1$: $D_v f(0,0) = 0$. If f is differentiable at $(0,0)$ then $Df(0,0) = 0$. Thus f is diff. at $(0,0)$ iff

$$\lim_{\vec h\to 0} \frac{f(\vec h) - f(0) - Df(0,0)h^T}{\|h\|} = 0$$

$$\boxed{\text{iff} \quad \lim_{\vec h\to 0} \frac{h_1^2 |h_2|^\alpha}{\left(h_1^2 + |h_2|^3\right)\sqrt{h_1^2 + h_2^2}} = 0}$$

~~If $\alpha > 1$ write $\alpha = 1+s$, $s>0$~~

~~$\lim_{\vec h\to 0} h_1^2 |h|$~~

Write

$$\frac{h_1^2 |h_2|^\alpha}{(h_1^2+|h_2|^3)\sqrt{h_1^2+h_2^2}} = \frac{h_1^{2-s_1}|h_2|^{\alpha-s_2}}{h_1^2+|h_2|^3} \; \frac{h_1^{s_1}|h_2|^{s_2}}{\sqrt{h_1^2+h_2^2}}$$

Young's Ineq

$$\leq c\left(\frac{h_1^{(2-s_1)p_1} + |h_2|^{(\alpha-s_2)q_1}}{h_1^2+|h_2|^3}\right)\left(\frac{h_1^{s_1 p_2} + |h_2|^{s_2 q_2}}{\sqrt{h_1^2+h_2^2}}\right)$$

where $s_1 > 0$, $s_2 > 0$ $\qquad \frac{1}{p_1}+\frac{1}{q_1} = \frac{1}{p_2}+\frac{1}{q_2} = 1$

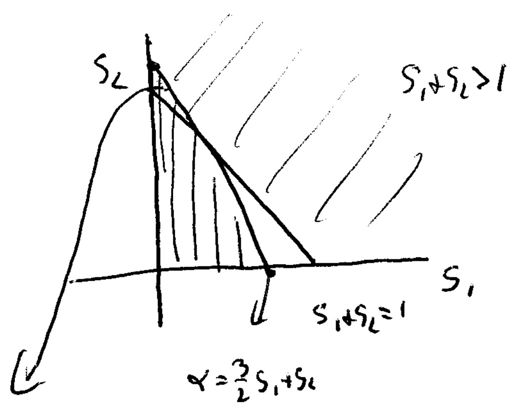
We need
$$(2-s_1)p_1 > 2 \qquad s_1 p_2 > 1$$
$$(\alpha-s_2)q_1 > 3 \quad \text{and} \quad s_2 q_2 > 1$$

for the limit to be zero. Equivalently

$$1-\frac{s_1}{2} > \frac{1}{p_1} \qquad s_1 > \frac{1}{p_2}$$
$$\frac{\alpha-s_2}{3} > \frac{1}{q_1} \qquad s_2 > \frac{1}{q_2}$$

$$1-\frac{s_1}{2}+\frac{\alpha-s_2}{3} > \frac{1}{p_1}+\frac{1}{q_1} = 1 \qquad s_1+s_2 > \frac{1}{p_2}+\frac{1}{q_2} = 1$$

Must hold simultaneously
$$\boxed{\begin{array}{c} s_1+s_2 > 1 \\ \frac{\alpha}{3} > \frac{s_1}{2}+\frac{s_2}{3} \end{array}}$$



If $\alpha > 1$ then

there is a wedge of choices for s_1, s_2 s.t. the limit is zero.

HWS NB #2

Consider $\mathbb{R}^\infty = \{ \overbrace{(x_1, \cdots, x_n, \cdots)}^{\vec{x}} \mid x_i \in \mathbb{R} \text{ for } i=1,2,3,\ldots \}$

$$\|\vec{x}\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$$

The Space $\ell^2 = \{\vec{x} \in \mathbb{R}^\infty \mid \|x\|_2 < \infty\}$ is infinite dimensional

show that $S \equiv \{\vec{x} \in \ell^2 \mid \|x\|_2 = 1\}$ is closed, bounded,

and not sequentially compact.

If $\{\vec{X}_n\} \subset S$ and $\vec{X}_n \to \vec{x}$ in ℓ^2 then

$$\|\vec{x}_n - \vec{x}\|_2 \to 0$$

$$\Rightarrow \left| \|\vec{x}_n\| - \|\vec{x}\| \right| \to 0 \quad \Rightarrow \quad \|\vec{x}\| = 1 \Rightarrow \vec{x} \in S.$$

So S is closed.

Clearly S is bdd, by 1.

Let $\vec{a}_n = \cancel{\{} (0, \cdots, 0, \underset{n\text{th spot}}{1}, 0, \cdots, 0 \cdots)$

then $\|\vec{a}_n\|_2 = 1$ so $\vec{a}_n \in S$, but if

$n \neq \ell$ then $\|\vec{a}_n - \vec{a}_\ell\|_2 = \sqrt{2}$

So if $\{\vec{a}_{n_\ell}\}$ is a convergent subsequence we have

a contradiction as $\vec{a}_{n_\ell} \to \vec{a}$ $\Rightarrow \sqrt{2} = \|\vec{a}_{n_{\ell_1}} - \vec{a}_{n_{\ell_2}}\| \leq \|\vec{a} - \vec{a}_{n_{\ell_1}}\| + \|\vec{a} - a_{n_{\ell_2}}\|$

$\leq 2\varepsilon$

$\forall \varepsilon > 0.$