9.4.8 Suppose that $f: D \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous on D. Show that f has a continuous extension to $E \equiv \overline{D}$.

Define f on E

(a) Let $x \in E$, then there exists $\{x_n\}_1^\infty \subset D \ni x_n \to x$.

Since $x_n \to x$ the sequence $\{x_n\}_{n=1}^\infty$ is Cauchy. Since f is uniformly cont. $\forall \varepsilon > 0 \ \exists \delta \ni . \ \|x-y\| < \delta \quad x, y \in D \Rightarrow$

$\|f(x) - f(y)\| < \varepsilon$. Since $\{x_n\}_{n=1}^\infty$ is Cauchy $\exists N > 0 \ni$.

$n, m \geq N \Rightarrow \|x_n - x_m\| < \delta \Rightarrow \|f(x_n) - f(x_m)\| < \varepsilon \Rightarrow \{f(x_n)\}_{n=1}^\infty$ is Cauchy in $\mathbb{R}^m$. Since $\mathbb{R}^n$ is complete $\exists y \in \mathbb{R}^m \ni$.

$$f(x_n) \to y, \quad \text{we \underline{define}} \quad f(x) := y.$$

(b). Show f is well-defined. Suppose $x \in E$, $\{x_n\} \subset D \to x$
$\{\tilde{x}_n\} \subset D \to x$

then we have shown $f(x_n) \to y$ and $f(\tilde{x}_n) \to \tilde{y} \quad \exists y, \tilde{y} \in \mathbb{R}^m$

Does $y = \tilde{y}$? Since $x_n \to x$, $\tilde{x}_n \to x$ then $\forall \varepsilon > 0 \ \exists N \ni$.

$n \geq N \Rightarrow \|x_n - x\| < \varepsilon, \ \|\tilde{x}_n - x\| < \varepsilon \Rightarrow$

$$\boxed{\begin{aligned} \|x_n - \tilde{x}_m\| &\leq \|x_n - x + x - \tilde{x}_m\| \\ &\leq \|x_n - x\| + \|x - \tilde{x}_m\| \\ \forall n, m \geq N \quad &\leq 2\varepsilon \end{aligned}}$$

in particular, take $2\varepsilon = \delta$ in continuity and
$\|f(x_n) - f(\tilde{x}_m)\| < \varepsilon \quad \forall n, m \geq N_{\delta(\varepsilon)}$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{m\to\infty} f(\tilde{x}_m)$$

$$y = \tilde{y}$$

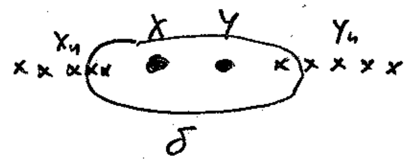
So $f(x)$ is \underline{independent} of the choice of $\{x_n\} \subset D \to x$.

(c) Show f is cont. on E

Fix $\varepsilon > 0$, show $\exists\ \delta > 0$ s.t. $\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$.

Let $\{x_n\} \subset D \to x$

$\{y_n\} \subset D \to y$



then $f(x_n) \to f(x)$

$f(y_n) \to f(y)$

$\forall \delta > 0\ \exists N > 0$ s.t. $n \geq N \Rightarrow$ $|x_n - x| < \delta$

$|y_m - y| < \delta$

$\Rightarrow |x_n - y_m| < |x_n - x + x - y + y - y_m|$

$\leq \|x_n - x\| + \|x - y\| + \|y - y_m\| \leq 3\delta$

Take δ small enough and $\|x_n - y_m\| \leq 3\delta \Rightarrow \|f(x_n) - f(y_m)\| < \varepsilon$

$\forall n \geq N_\delta$

$$\Rightarrow \lim_{n,m \to \infty} \|f(x_n) - f(y_m)\| < \varepsilon$$

$$\|f(x) - f(y)\| < \varepsilon \quad \therefore$$