M421 HW 4

Due Friday Oct. 26

From Wade

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Non-book Exercises

1) For two sets $A, B \subset \mathbf{R}^n$ define

$$\operatorname{dist}(A,B) = \inf_{\vec{x}\in A, \vec{y}\in B} \|\vec{x}-\vec{y}\|.$$

(a) Show that if $E \subset \mathbf{R}^n$ is closed and $K \subset \mathbf{R}^n$ is compact then

$$E \cap K = \emptyset, \qquad \Longleftrightarrow \qquad \operatorname{dist}(E,K) > 0.$$

(b) Find two closed sets E_1 and E_2 in $\mathbf{R}^2$ such that $\operatorname{dist}(E_1, E_2) = 0$ but $E_1 \cap E_2 = \emptyset$.

2) Suppose that $E \subset \mathbf{R}^n$ is connected and $E \subset A \subset \overline{E}$. Prove that A is connected.

3) Let f be defined on $\mathbf{R}^2$ by

$$f(x,y) = \frac{x|y|^\alpha}{x^2+y^2}.$$

For which $\alpha > 0$ is it true that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

9.2) 3a) Prove each of the following has a limit as $\vec{x} \to 0$

a) $f(x,y) = \dfrac{x^3 - y^3}{x^2+y^2} \qquad \vec{x} \neq 0$

Proof

$$|f(x,y)| \le \frac{|x^3|}{x^2+y^2} + \frac{|y^3|}{x^2+y^2} \le \frac{|x^3|}{x^2} + \frac{|y^3|}{y^2} \le |x| + |y| \le 2\|\vec{x}\|$$

drop

So if $\|\vec{x}\| < \varepsilon/2$ then $|f(\vec{x})| < \varepsilon \Rightarrow \lim_{\vec{x}\to 0} f(\vec{x}) = 0$.

b) $f(x,y) = \dfrac{|x|^\alpha y^4}{x^2+y^2}$

proof

$$|f(x,y)| \le \frac{|x|^\alpha y^4}{x^2+y^2} \le |x|^\alpha y^2$$

drop

So if $\|\vec{x}\| < \varepsilon$ then $|x| \le \varepsilon$, $|y| \le \varepsilon$, hence if $\alpha > 0$ then

$$|f(x,y)| \le \varepsilon^{2+\alpha} \to 0 \quad \text{as } \varepsilon \to 0$$

9.3) $\boxed{5}$ Let $B^{\text{closed}} \subset \mathbb{R}^n$ and $f: B \to \mathbb{R}^m$ prove

f is cont. on B iff $f^{-1}(E)$ is closed $\forall\ E^{\text{closed}} \subset \mathbb{R}^m$

$\boxed{\Rightarrow}$ Let $E \subset \mathbb{R}^n$ then

$$f^{-1}(E) = \{x \in B \mid f(x) \in E\}$$

$$\left(f^{-1}(E)\right)^c = \{x \in B \mid f(x) \notin E\} \cup B^c$$

$$\boxed{\left(f^{-1}(E)\right)^c = f^{-1}(E^c) \cup B^c}$$

If $O \subset \mathbb{R}^n$ is open then f cont. $\Rightarrow$

$f^{-1}(O)$ is open.

Let $E \subset \mathbb{R}^m$ be closed, then E^c is open

$\Rightarrow$ $\underline{f^{-1}(E^c)}$ is open

$\underline{B \text{ closed}} \Rightarrow B^c$ open

So $\left(f^{-1}(E)\right)^c = f^{-1}(E^c) \cup B^c$ = open $\cup$ open = open

$\Rightarrow$ $\underline{f^{-1}(E) = \text{closed}}$

$\boxed{\Leftarrow}$ Suppose E closed $\Rightarrow f^{-1}(E)$ is closed but f is <u>not</u> cont. on B.

then f not cont. $\Rightarrow \exists \{\vec{x}_n\} \subset B \ni \vec{x}_n \to \vec{x} \in B$ but $f(\vec{x}_n) \not\to f(\vec{x})$

Let $E = \overline{\{f(x_n) \mid n = 1, 2, \ldots\}}$ closure

then $\underline{f(x) \notin E}$. But $f^{-1}(E) \supset \{x_n\}_{n=1}^{\infty}$ and $f^{-1}(E)$ is closed

so $\vec{x} \in f^{-1}(E)$ since $\vec{x}_n \to \vec{x}$. $\Rightarrow \underline{f(\vec{x}) \in E}$ a contradiction

9.4| #2) Let $A, B \subset \mathbb{R}^n$ be compact, show $A \cup B$, $A \cap B$ are compact

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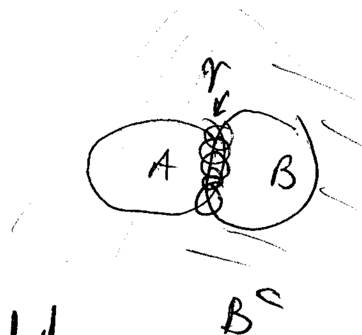
Let $\mathcal{V}$ be an open cover of $A \cup B$, then

$\mathcal{V}$ covers A and $\mathcal{V}$ covers B $\Rightarrow$

$\exists$ finite subcover $\{V_1, \dots V_{N_A}\}$ which covers A

$\{V_{N_A+1}, \dots, V_{N_A+N_B}\}$ which covers B

and $\{V_1, \dots, V_{N_A+N_B}\}$ covers $A \cup B$.



If $\mathcal{V}$ is an open cover of $A \cap B$ then

$\tilde{\mathcal{V}} = \mathcal{V} \cup \{B^c\}$ is an open cover of A

so $\underset{A \text{ compact} \Rightarrow \exists}{\wedge}$ $\{V_1, \dots, V_N, B^c\}$ is a finite subcover of A

Since $A \cap B \subset A$ and $(A \cap B) \cap B^c = \phi$ $\Rightarrow$ $\{V_1, \dots, V_N\}$ is a finite subcover of $A \cap B$ from $\mathcal{V}$.

Non-book HW4 #1

For $A, B \subset \mathbb{R}^n$ define

$$\text{dist}(A,B) = \inf_{\substack{\vec{x} \in A \\ \vec{y} \in B}} \|\vec{x} - \vec{y}\|$$

Show that if $E \subset \mathbb{R}^n$ is closed and $K \subset \mathbb{R}^n$ is compact then

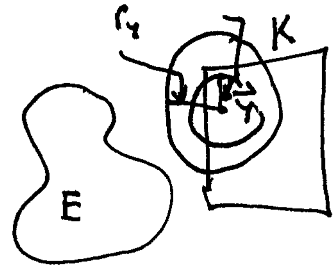
$$E \cap K = \emptyset \quad \text{iff} \quad \text{dist}(E,K) > 0.$$

Method I

$\boxed{\Rightarrow}$ Suppose $E \cap K = \emptyset$, show $\exists \alpha > 0$ s.t. $\text{dist}(E,K) = \alpha$.

Let $\vec{y} \in K$, then $\vec{y} \notin E \Rightarrow \vec{y} \in E^c$ which is open,

so $\exists r_y > 0$ s.t. $B_{r_y}(\vec{y}) \subset E^c$.

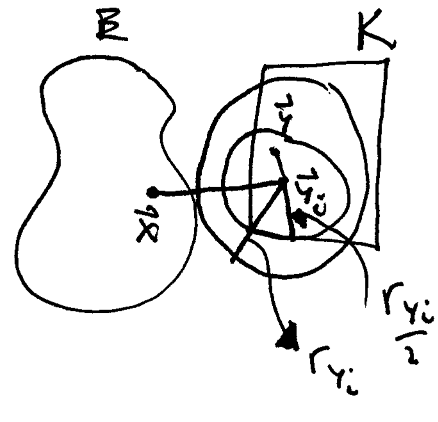


Let $\tilde{B}_y = B_{\left(\frac{r_y}{2}\right)}(\vec{y}) \subset B_{r_y}(\vec{y}) \subset E^c$

Then $\mathcal{V} = \{\tilde{B}_y \mid \vec{y} \in K\}$ is an open cover of K.

Let $\{B_{\vec{y}_1}, \dots, B_{\vec{y}_N}\}$ be a finite subcover

let $\alpha = \min_{i=1,\dots,N}\left\{\frac{r_{\vec{y}_i}}{2}\right\} > 0$



then $\vec{y} \in K$ and $\vec{x} \in E$

$\Rightarrow \vec{y} \in B_{\vec{y}_i}$ for some $i \in \{1, \dots, N\}$

$\Rightarrow \|\vec{x} - \vec{y}\| = \|x - \vec{y}_i + \vec{y}_i - \vec{y}\| \geq \big|\|\vec{x} - \vec{y}_i\| - \|\vec{y}_i - \vec{y}\|\big|$

$\geq r_{y_i} - \frac{r_{y_i}}{2} = \frac{r_{y_i}}{2} \geq \alpha.$

$\boxed{\text{Method 2}}$ If $\operatorname{dist}(E, K) = 0$

then $\exists \{\vec{x}_n\} \subset E$, $\{\vec{y}_n\} \subset K$ $\ni$.

$$\|\vec{x}_n - \vec{y}_n\| < 1/n$$

Since K is compact $\vec{y}_{n_k} \to \vec{y} \in K$

$\forall \varepsilon > 0$ $\exists N_\varepsilon > 0$ $\ni$.

$$\|\vec{x}_n - \vec{y}\| \le \|\vec{x}_n - \vec{y}_n\| + \|\vec{y}_n - \vec{y}\|$$

$$\le \frac{1}{n} + \frac{\varepsilon}{2} \qquad \forall n \ge N_\varepsilon$$

$$\le \varepsilon \qquad \forall n \ge \operatorname{Max}\{\tfrac{1}{\varepsilon}, N_\varepsilon\}$$

$\Rightarrow \vec{x}_n \to \vec{y}$

but E closed $\Rightarrow \vec{y} \in E$

$\Rightarrow \vec{y} \in E \cap K$ $\#$ since $E \cap K = \phi$

HW #4
NB #2

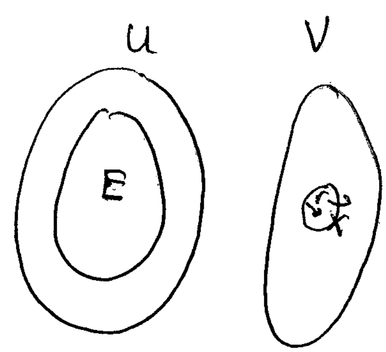
Suppose $E \subset \mathbb{R}^n$ is connected and $E \subset A \subset \bar{E}$.
Prove that A is connected

Suppose A is not connected, then $\exists\, U, V$ open $\ni$

$$U \cap A \neq \phi \qquad U \cup V \supset A$$
$$V \cap A \neq \phi \qquad U \cap V = \phi$$

but $E \subset A \subset U \cap V$ so either $U \supset E$ or $V \supset E$

since E is connected



WLOG $U \supset E$. But $\exists\, \vec{x} \in V \cap A \Rightarrow \vec{x} \in \bar{E}$

$\Rightarrow \exists \{\vec{x}_n\} \subset E \ni \vec{x}_n \to \vec{x}$

but $\vec{x} \in V \Rightarrow \exists\, r > 0 \ni B_r(\vec{x}) \subset V$

so $\vec{x}_n \in E \Rightarrow \vec{x}_n \notin B_r(\vec{x})$

$\Rightarrow \|\vec{x}_n - \vec{x}\| > r \Rightarrow \vec{x}_n \not\to \vec{x}$ #.

Non-book

#3) Let f be defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$f(x,y) = \frac{x|y|^\alpha}{x^2+y^2}$$

for which $\alpha > 0$ is it true that

$$\lim_{\vec{x} \to 0} f(x,y) = 0 \quad ?$$

By Young's inequality, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$x|y|^\alpha \le \frac{|x|^p}{p} + \frac{|y|^{\alpha q}}{q}$$

We want to choose p, q such that

$$\begin{aligned} p &> 2 \\ \alpha q &> 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\frac{1}{p} < \frac{1}{2} \\ &\frac{1}{\alpha q} < \frac{1}{2} \quad \text{or} \quad \frac{1}{q} < \frac{\alpha}{2} \end{aligned}$$

but $\frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2} + \frac{\alpha}{2}$.

These two statements are compatible only if

$$\frac{1}{2} + \frac{\alpha}{2} > 1 \quad \text{or} \quad \boxed{\alpha > 1}$$

For $\alpha > 1$, let $s = \text{Min}\{p-2, \alpha q - 2\} > 0$

then
$$|xy^\alpha| \le \frac{|x|^p}{p} + \frac{|y|^{\alpha q}}{q} \le \frac{1}{p}|x|^2|x|^s + \frac{1}{q}|y|^2|y|^s$$

Fix $\varepsilon > 0$, and let $\|\vec{x}\| < \delta$ where δ will be choosen later.

$$|xy^\alpha| \le \frac{1}{p}|x|^2|x|^s + \frac{1}{q}|y|^2|y|^s \le \frac{1}{p}\delta^s|x|^2 + \frac{1}{q}\delta^s|y|^2$$
$$\le c\delta^s\left(|x|^2 + |y|^2\right)$$

where $c = \max\left(\frac{1}{p}, \frac{1}{q}\right)$

$$\therefore \frac{|x|y|^\alpha|}{x^2+y^2} \le \frac{c\delta^s\left(|x|^2+|y|^2\right)}{x^2+y^2} \le c\delta^s$$

Let $\delta = \left(\frac{\varepsilon}{c}\right)^{1/s}$, then $\|\vec{x}\| < \delta \Rightarrow$

$$\left|\frac{x|y|^\alpha}{x^2+y^2}\right| < \varepsilon$$

$$\Rightarrow \lim_{\vec{x}\to 0} \frac{x|y|^\alpha}{x^2+y^2} = 0 \quad \text{if} \quad \underline{\alpha > 1}$$

If $0 < \alpha \le 1$

Let $y = x$, and $x \to 0$

$$\lim_{x\to 0^+} f(x,x) = \frac{|x|^{1+\alpha}}{2x^2} = \frac{1}{2}|x|^{\alpha-1} = \begin{cases} +\infty & \alpha < 1 \\ \frac{1}{2} & \alpha = 1 \end{cases}$$

Since this directional limit does not approach zero, the limit cannot be zero

$$\lim_{\vec{x}\to 0} f(x,y) \ne 0 \quad \text{for } 0 < \alpha \le 1$$