8.3 #3

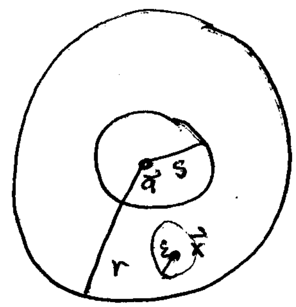
Let $n \in \mathbb{N}$, $\vec{a} \in \mathbb{R}^n$ and $s, r \in \mathbb{R}$ with $0 < s < r$. show that

$$V = \{ x \in \mathbb{R}^n \mid s < \|x - a\| < r \}$$

is open

Let $\vec{x} \in V$ and set

$$\varepsilon = \min\left\{ \frac{r - \|x-a\|}{2}, \frac{\|x-a\| - s}{2} \right\}$$



show that $B_\varepsilon(\vec{x}) \subset V$.

If $\vec{y} \in B_\varepsilon(\vec{x})$ then

$$\|\vec{a} - \vec{y}\| = \|\vec{a} - \vec{x} + \vec{x} - \vec{y}\| \overset{\Delta \text{ ineq}}{\le} \|\vec{a} - \vec{x}\| + \|\vec{x} - \vec{y}\|$$
$$\le \|\vec{a} - \vec{x}\| + \varepsilon \le \|\vec{a} - \vec{x}\| + \frac{r - \|\vec{x} - \vec{a}\|}{2}$$
$$\le \frac{\|\vec{x} - \vec{a}\| + r}{2} < r$$

similarly

$$\|\vec{a} - \vec{y}\| \overset{\text{reverse } \Delta \text{ ineq}}{\ge} \left| \|x - \vec{a}\| - \|\vec{x} - \vec{y}\| \right| \ge \|\vec{x} - \vec{a}\| - \varepsilon \ge \|\vec{x} - \vec{a}\| - \frac{\|x - \vec{a}\| - s}{2}$$
$$= \frac{\|\vec{x} - \vec{a}\| + s}{2} > s$$

thus $\vec{y} \in V$ and $B_\varepsilon(\vec{x}) \subset V$

hence V is open

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5) Show that if E is closed in $\mathbb{R}^n$ and $\vec{a} \notin E$ then

$$\inf_{x \in E} \|x-a\| > 0$$

Since E is closed, E^c is open

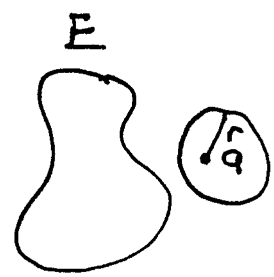
So $\vec{a} \in E^c \Rightarrow \exists\, r>0 \ni B_r(\vec{a}) \subset E^c$.

If $\vec{x} \in E$, then $\vec{x} \notin E^c \Rightarrow \vec{x} \notin B_r(\vec{a})$

$$\Rightarrow \|\vec{x}-\vec{a}\| \geq r$$

This is true $\forall\ \vec{x} \in E$ so

$$\inf_{\vec{x} \in E} \|\vec{x}-\vec{a}\| \geq r > 0$$



8.4] #9) (b) show $\exists$ sets $A, B \subset \mathbb{R}^n$ $\ni$.

$$\overline{A \cap B} \neq \overline{A} \cap \overline{B}$$

Let $A = \mathbb{Q} \cap [0,1]$

$B = [0,1] \setminus \mathbb{Q}$

$\mathbb{Q}$ = Rationals

$A \cap B = \phi$ and $\overline{A \cap B} = \phi$

but $A \cup B = [0,1]$

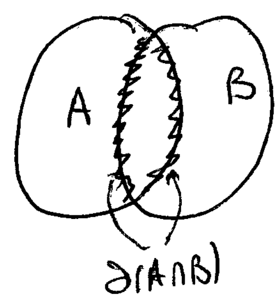
so $\overline{A \cup B} = [0,1] \neq \phi$.

8.4 #10a) Show that $\partial(A\cap B)\cap(A^c\cup(\partial B)^c)\subset\partial A$

Suppose $\vec{x}\in\partial(A\cap B)$ and $\vec{x}\in A^c\cup(\partial B)^c$

and $\vec{x}\notin\partial A$, look for a contradiction.

If $\vec{x}\notin\partial A$ then $\vec{x}\in A^o$ or $\vec{x}\in(A^c)^o$



Case 1 If $\vec{x}\in(A^c)^o$ then $\vec{x}\notin\partial(A\cap B)$.

proof

If $\vec{x}\in\partial(A\cap B)$ then $\exists\ \{x_n\}\subset(A\cap B)$

$\ni.\ \vec{x}_n\to\vec{x}$, but if $\vec{x}\in(A^c)^o$ then $\exists\ \varepsilon>0$

$\ni.\ B_\varepsilon(\vec{x})\subset A^c \Rightarrow \|\vec{x}_n-\vec{x}\|>\varepsilon$ since $\vec{x}_n\in A$

$\Rightarrow \vec{x}_n\not\to\vec{x}_n$ #.

Hence $(A^c)^o\cap\partial(A\cap B)=\phi$

Case 2 If $\vec{x}\in A^o$ and $\vec{x}\in\partial(A\cap B)$ then $\vec{x}\in\partial B$.

proof

If $\vec{x}\in\partial(A\cap B)$ then $\exists\ \{x_n\}\subset(A\cap B)\ \ni.\ \vec{x}_n\to\vec{x}$

$\exists\ \{y_n\}\subset(A\cap B)^c=A^c\cup B^c\ \ni.\ \vec{y}_n\to\vec{x}$

but $\vec{x}\in A^o$ so no sequence from A^c can converge to $\vec{x}$.

So, passing to a subsequence, $\vec{y}_{n_k}\subset B^c$ and $\vec{y}_{n_k}\to\vec{x}$

but $\vec{x}_n\subset(A\cap B)\subset B$ $\quad\vec{x}_n\to\vec{x}$

so $\vec{x}\in\partial B$.

Finally if $\vec{x}\in A^o$ and $\vec{x}\in\partial(A\cap B)$ and $\vec{x}\in A^c\cup(\partial B)^c$ then we have a contradiction since the first two $\Rightarrow\vec{x}\in\partial B$ so $\vec{x}\in A^c\cup(\partial B)^c\Rightarrow\vec{x}\in A^c$, but $\vec{x}\in A^o$ and $A^c\cap A^o=\phi$ #.

9.1 #8 (a) Prove every closed ball in $\mathbb{R}^n$ is sequentially compact

If $\{x_n\}_{n=1}^{\infty} \subset E$ and E is a closed ball about $\vec{a}$ of radius r then

$$\|\vec{x}_n - \vec{a}\| \le r \Rightarrow \|\vec{x}_n\| \le \|\vec{a}\| + r \quad \forall n$$
$$\Rightarrow \{\vec{x}_n\} \text{ is bdd.}$$

by Bolzano-Weierstrass $\{\vec{x}_n\}$ has a subsequence $\{\vec{x}_{n_k}\}_{k=1}^{\infty}$ which converges to $\vec{x} \in \mathbb{R}^n$. Since E is closed $\vec{x} \in E$.

Thus E is sequentially compact.

(b) Prove $\mathbb{R}^n$ is not sequentially compact.

Let $\vec{x}_k = (k, 0, 0, \ldots, 0) \in \mathbb{R}^n \quad k = 1, 2, 3, \ldots$

then $\|\vec{x}_k - \vec{x}_j\| = k - j \ge 1$ if $k \ne j$

If $\{\vec{x}_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{\vec{x}_{k_j}\}$

$$\vec{x}_{k_j} \to \vec{x} \quad \text{then } \exists N > 0 \text{ s.t. } j > N \Rightarrow$$
$$\|x_{k_j} - \vec{x}\| < 1/4$$

so if $j_1, j_2 > N$ then

$$\|x_{k_{j_1}} - x_{k_{j_2}}\| \le \|\vec{x}_{k_{j_1}} - \vec{x}\| + \|\vec{x}_{k_{j_2}} - \vec{x}\|$$
$$\le 1/4 + 1/4 = 1/2$$

but $\|x_{k_{j_1}} - x_{k_{j_2}}\| = |k_{j_1} - k_{j_2}| \ge 1 \not< 1/2$ #.