M421 HW 2

Due Friday Sept. 22

From Wade

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Non-book Exercises

1) For $n = 1, 2, \ldots$, define $g_n(x) = 2xne^{-nx^2}$ for $x \in [0, 1]$.

(a) Show that $\forall x \in [0, 1]$, $\lim_{n\to\infty} g_n(x) = 0$.

(b) Using the Fundamental Theorem of Calculus to evalutate the integrals on the left, show that

$$\lim_{n\to\infty} \int_0^1 g_n(x)dx \neq \int_0^1 \left(\lim_{n\to\infty} g_n(x)\right) dx.$$

2) Show that if $f \in I[a, b]$ then $g(x) \equiv \sin(f(x)) \in I[a, b]$.

5.2| #3) Prove that if $f \in \mathscr{R}[0,1]$ and $\beta > 0$ then

$$\lim_{n\to\infty} n^{\alpha} \int_0^{n^{-\beta}} f(x)\,dx = 0$$

$\forall\ \alpha < \beta$.

Since $f \in \mathscr{R}[a,b]$, f is bdd on $[a,b]$, i.e. $\exists M > 0$ s.t.

$|f(x)| \leq M \quad \forall x \in [a,b]$.

Comparison principle

So $$\left| \int_0^{n^{-\beta}} f(x)\,dx \right| \leq \int_0^{n^{-\beta}} |f(x)|\,dx \leq \int_0^{n^{-\beta}} M\,dx = M n^{-\beta}$$

hence $$\left| n^{\alpha} \int_0^{n^{-\beta}} f(x)\,dx \right| \leq M n^{\alpha-\beta}$$

So $\alpha - \beta < 0 \Rightarrow \lim_{n\to\infty} n^{\alpha} \int_0^{n^{-\beta}} f(x)\,dx = 0$.

5.2 | 5a) Suppose $g_n \geq 0$ and $g_n \in \mathcal{R}[a,b]$, and

$$\lim_{n\to\infty} \int_a^b g_n(x)dx = 0.$$

Show that if $f \in \mathcal{R}[a,b]$ then

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0.$$

Since $f \in \mathcal{R}[a,b]$, f is bdd $\Rightarrow \exists M>0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

$$0 \leq \left|\int_a^b f(x)g_n(x)dx\right| \overset{\text{Thm 5.22}}{\leq} \int_a^b |f(x)|\, g_n(x)\, dx \quad g_n \geq 0$$

compairison principle

$$\leq \int_a^b M g_n(x)dx$$

$$\leq M\int_a^b g_n(x)dx \to 0 \quad \text{as } n\to\infty$$

By the Squeeze Thm

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)dx = 0.$$

[redacted]

5.2 | 5a) Suppose $g_n \geq 0$ and $g_n \in \mathscr{R}[a,b]$, and

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0.$$

Show that if $f \in \mathscr{R}[a,b]$ then

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)\,dx = 0.$$

Since $f \in \mathscr{R}[a,b]$, f is bdd $\Rightarrow \exists\, M>0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

Thm 5.22 $\qquad g_n \geq 0$

$$0 \leq \left| \int_a^b f(x)g_n(x)\,dx \right| \leq \int_a^b |f(x)|\, g_n(x)\,dx$$

comparison principle

$$\leq \int_a^b M g_n(x)\,dx$$

$$\leq M \int_a^b g_n(x)\,dx \to 0 \quad \text{as } n \to \infty$$

By the squeeze Thm

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)\,dx = 0.$$

5.2 #8 Let $f \in \mathcal{C}[a,b]$ and $M = \sup_{x\in[a,b]} |f(x)| > 0$

(a) Let $p>0$, show $\forall \varepsilon > 0$ $\exists$ interval $I \subset [a,b]$ $\ni$

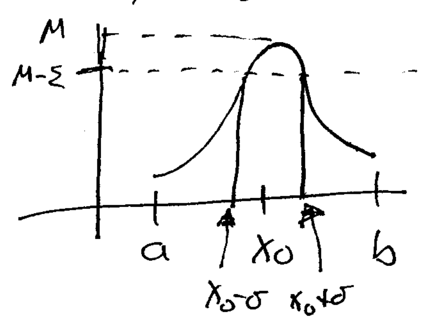
$$(M-\varepsilon)^p |I| \le \int_a^b |f(x)|^p dx \le M^p(b-a)$$

Since $|f(x)|^p \le M^p$ $\forall x \in [a,b]$, the second inequality follows from the comparison principle (Thm 5.21).

For the first inequality, let $x_0 \in [a,b]$ be such that $|f(x_0)| = M$ (f cont. $\Rightarrow$ f attains its Maxima).

Since f is cont, $\forall \varepsilon > 0$ $\exists \delta > 0$ $\ni$ $|f(x) - \underset{M}{\underset{\shortparallel}{f(x_0)}}| < \varepsilon$ $\forall$ $|x - x_0| < \delta$

$$\Rightarrow |f(x)| \ge M - \varepsilon \quad \forall x \in I \equiv [x_0 - \delta, x_0 + \delta]$$



this implies that

$$|f(x)|^p \ge (M-\varepsilon)^p \quad \text{on } I.$$

Thus, by the comparison Principle

$$(M-\varepsilon)^p |I| \le \int_I |f(x)|^p dx \le \int_a^b |f(x)|^p dx$$

where the second inequality follows since $|f(x)| \ge 0$.

(b) Show that $\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx\right)^{1/p} = M$

Take the p'th root of the inequality in (a)

I depends upon ε but not on p

$$(M-\varepsilon)|I|^{1/p} \le \left(\int_a^b |f(x)|^p dx\right)^{1/p} \le M(b-a)^{1/p}$$

Since $\lim_{p\to\infty} |I|^{1/p} = \lim_{p\to\infty} (b-a)^{1/p} = 1$, it follows that

$$M - \varepsilon \le \lim_{p\to\infty} \left(\int_a^b |f(x)|^p\right)^{1/p} \le M$$

since $\varepsilon > 0$ was arbitrary, the result follows.

5.3) 8) If $f \in \mathcal{C}[a,b]$ and $\exists\ \alpha \neq \beta \ni$.

$$\alpha \int_a^c f(x)\,dx + \beta \int_c^b f(x)\,dx = 0$$

$\forall\, c \in [a,b]$, prove $f(x) = 0 \quad \forall x \in [a,b]$.

Since f is continuous the two indefinite integrals are differentiable in c. Moreover

$$\frac{d}{dc}\left[\alpha \int_a^c f(x)\,dx + \beta \int_c^b f(x)\,dx\right] = 0$$

by the FToC (Fund. Thm of Calculus)

$$\alpha f(c) - \beta f(c) = 0$$

$$(\alpha - \beta) f(c) = 0$$

Since $\alpha \neq \beta$

$$f(c) = 0 \quad \forall c \in [a,b].$$

Non book #2 | Show that if $f \in I[a,b]$ then
$g(x) \equiv \sin(f(x)) \in I[a,b]$

Since $f \in I[a,b]$, $\forall \varepsilon > 0$ $\exists P \in \mathcal{P}[a,b]$ $\ni$.

$$U(f,P) - L(f,P) = \sum_{j=0}^{n} \left(M_j^f - m_j^f\right)(x_{j+1} - x_j) < \varepsilon$$

where $M_j^f = \sup_{[x_j, x_{j+1}]} f(x)$, $m_j^f = \inf_{[x_j, x_{j+1}]} f(x)$

Now $\sin(x) - \sin(y) = \cos(\alpha)(x-y)$ Mean Value Thm for some $x < \alpha < y$

so $|\sin(x) - \sin(y)| \le |\cos\alpha|\,|x-y| \le |x-y|$

Fix $\varepsilon > 0$

For the Partition $P = \{x_0, \dots, x_n\}$, there exists $x_i^+, x_i^- \in [x_i, x_{i+1}]$

$\ni$. $0 < M_i^g - g(x_i^+) < \varepsilon$

$0 < g(x_i^-) - m_i^g < \varepsilon$

where $M_i^g = \sup_{[x_i, x_{i+1}]} g(x)$, $m_i^g = \inf_{[x_i, x_{i+1}]} g(x)$

so

$$0 \le M_i^g - m_i^g < g(x_i^+) - g(x_i^-) + 2\varepsilon$$

but $|g(x_i^+) - g(x_i^-)| = |\sin(f(x_i^+)) - \sin(f(x_i^-))|$

$$\le |f(x_i^+) - f(x_i^-)| \le M_i^f - m_i^f$$

That is

$$M_i^g - m_i^g \le M_i^f - m_i^f + 2\varepsilon$$

$$\Rightarrow U(g,P) - L(g,P) = \sum \left(M_i^g - m_i^g\right)(x_{j+1} - x_j)$$

$$\le \sum \left(M_i^f - m_i^f + 2\varepsilon\right)(x_{i+1} - x_j)$$

$$\le U(f,P) - L(f,P) + 2\varepsilon(b-a)$$

$$\le \varepsilon + 2\varepsilon(b-a) \le \varepsilon(1 + 2(b-a))$$ $\boxtimes$

A bit slicker way to get the main idea

~~Result~~ Realize that

not trivial but true

$$M_i^g - m_i^g = \sup_{x,y\in[x_i,x_{i+1}]} \big(g(x) - g(y)\big) = \sup_{x,y\in[x_i,x_{i+1}]} |g(x) - g(y)|$$

but

$$|g(x) - g(y)| = |\sin(f(x)) - \sin(f(y))|$$

Mean Value Thm

$$\leq |f(x) - f(y)|$$

so

$$M_i^g - m_i^g = \sup_{x,y\in[x_i,x_{i+1}]} |g(x) - g(y)| \leq \sup_{x,y\in[\,]} |f(x) - f(y)|$$

$$\leq M_i^f - m_i^g$$