EXERCISES 10.4

1. A single equation involving the coordinates (x, y, z) need not always represent a two-dimensional "surface" in $\mathbb{R}^3$. For example, $x^2 + y^2 + z^2 = 0$ represents the single point $(0, 0, 0)$, which has dimension zero. Give examples of single equations in x, y, and z that represent
   (a) a (one-dimensional) straight line,
   (b) the whole of $\mathbb{R}^3$,
   (c) no points at all (i.e., the empty set).

In Exercises 2–9, find equations of the planes satisfying the given conditions.

2. Passing through $(0, 2, -3)$ and normal to the vector $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
3. Passing through the origin and having normal $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
4. Passing through $(1, 2, 3)$ and parallel to the plane $3x + y - 2z = 15$
5. Passing through the three points $(1, 1, 0)$, $(2, 0, 2)$, and $(0, 3, 3)$
6. Passing through the three points $(-2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$
7. Passing through $(1, 1, 1)$ and $(2, 0, 3)$ and perpendicular to the plane $x + 2y - 3z = 0$
8. Passing through the line of intersection of the planes $2x + 3y - z = 0$ and $x - 4y + 2z = -5$, and passing through the point $(-2, 0, -1)$
9. Passing through the line $x + y = 2$, $y - z = 3$, and perpendicular to the plane $2x + 3y + 4z = 5$
10. Under what geometric condition will three distinct points in $\mathbb{R}^3$ not determine a unique plane passing through them? How can this condition be expressed algebraically in terms of the position vectors, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, of the three points?
11. Give a condition on the position vectors of four points that guarantees that the four points are *coplanar*, that is, all lie on one plane.

Describe geometrically the one-parameter families of planes in Exercises 12–14. (λ is a real parameter.)

12. $x + y + z = \lambda$.
13. $x + \lambda y + \lambda z = \lambda$.
14. $\lambda x + \sqrt{1 - \lambda^2}\, y = 1$.

In Exercises 15–19, find equations of the line specified in vector and scalar parametric forms and in standard form.

15. Through the point $(1, 2, 3)$ and parallel to $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$
16. Through $(-1, 0, 1)$ and perpendicular to the plane $2x - y + 7z = 12$
17. Through the origin and parallel to the line of intersection of the planes $x + 2y - z = 2$ and $2x - y + 4z = 5$
18. Through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$
19. Through $(1, 2, -1)$ and making equal angles with the positive directions of the coordinate axes

In Exercises 20–22, find the equations of the given line in standard form.

20. $\mathbf{r} = (1 - 2t)\mathbf{i} + (4 + 3t)\mathbf{j} + (9 - 4t)\mathbf{k}$.
21. $\begin{cases} x = 4 - 5t \\ y = 3t \\ z = 7 \end{cases}$
22. $\begin{cases} x - 2y + 3z = 0 \\ 2x + 3y - 4z = 4 \end{cases}$
23. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, show that the equations

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

represent a line through P_1 and P_2.

24. What points on the line in Exercise 23 correspond to the parameter values $t = -1$, $t = 1/2$, and $t = 2$? Describe their locations.
25. Under what conditions on the position vectors of four distinct points P_1, P_2, P_3, and P_4 will the straight line through P_1 and P_2 intersect the straight line through P_3 and P_4 at a unique point?

Find the required distances in Exercises 26–29.

26. From the origin to the plane $x + 2y + 3z = 4$
27. From $(1, 2, 0)$ to the plane $3x - 4y - 5z = 2$
28. From the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$
29. Between the lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$

30. Show that the line $x - 2 = \dfrac{y+3}{2} = \dfrac{z-1}{4}$ is parallel to the plane $2y - z = 1$. What is the distance between the line and the plane?

In Exercises 31–32, describe the one-parameter families of straight lines represented by the given equations. (λ is a real parameter.)

31. $(1 - \lambda)(x - x_0) = \lambda(y - y_0)$, $z = z_0$.
32. $\dfrac{x - x_0}{\sqrt{1 - \lambda^2}} = \dfrac{y - y_0}{\lambda} = z - z_0$.
33. Why does the factored second-degree equation

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0$$

represent a pair of planes rather than a single straight line?

EXERCISES 10.5

Identify the surfaces represented by the equations in Exercises 1–16 and sketch their graphs.

1. $x^2+4y^2+9z^2=36$
2. $x^2+y^2+4z^2=4$
3. $2x^2+2y^2+2z^2-4x+8y-12z+27=0$
4. $x^2+4y^2+9z^2+4x-8y=8$
5. $z=x^2+2y^2$
6. $z=x^2-2y^2$
7. $x^2-y^2-z^2=4$
8. $-x^2+y^2+z^2=4$
9. $z=xy$
10. $x^2+4z^2=4$
11. $x^2-4z^2=4$
12. $y=z^2$
13. $x=z^2+z$
14. $x^2=y^2+2z^2$
15. $(z-1)^2=(x-2)^2+(y-3)^2$
16. $(z-1)^2=(x-2)^2+(y-3)^2+4$

Describe and sketch the geometric objects represented by the systems of equations in Exercises 17–20.

17. $\begin{cases} x^2+y^2+z^2=4 \\ x+y+z=1 \end{cases}$
18. $\begin{cases} x^2+y^2=1 \\ z=x+y \end{cases}$
19. $\begin{cases} z^2=x^2+y^2 \\ z=1+x \end{cases}$
20. $\begin{cases} x^2+2y^2+3z^2=6 \\ y=1 \end{cases}$
21. Find two one-parameter families of straight lines that lie on the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

22. Find two one-parameter families of straight lines that lie on the hyperbolic paraboloid $z=xy$.
23. The equation $2x^2+y^2=1$ represents a cylinder with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cylinder has circular cross-sections.
24. The equation $z^2=2x^2+y^2$ represents a cone with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cone has circular cross-sections. *Hint:* Do Exercise 23 first and use its result.

10.6 Cylindrical and Spherical Coordinates

Polar coordinates provide a useful alternative to plane Cartesian coordinates for describing plane regions with circular symmetry or bounded by arcs of circles centred at the origin and radial lines from the origin. Similarly, there are two commonly encountered alternatives to Cartesian coordinates in 3-space. They generalize plane polar coordinates to 3-space and are suitable for describing regions with cylindrical or spherical symmetry. We introduce these two coordinate systems here, but won't make much use of them until the latter part of Chapter 14 when we will learn how to integrate over such regions.

Cylindrical Coordinates

Among the most useful alternatives to Cartesian coordinates in 3-space is the coordinate systems that directly generalizes plane polar coordinates by replacing only the horizontal x and y coordinates with the polar coordinates r and θ, while leaving the vertical z coordinate untouched. This system is called **cylindrical coordinates**. Each point in 3-space has cylindrical coordinates $[r,\theta,z]$ related to its Cartesian coordinates (x,y,z) by the transformation

$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z.$$

Figure 10.38 shows how a point P is located by its cylindrical coordinates $[r,\theta,z]$ as well as by its Cartesian coordinates (x,y,z). Note that the distance from P to the z-axis is r, while the distance from P to the origin is

$$d=\sqrt{r^2+z^2}=\sqrt{x^2+y^2+z^2}.$$

THEOREM 7

If $\mathcal{A} = (a_{ij})_{i,j=1}^{n}$ is a real, symmetric matrix, then

(a) all the eigenvalues of $\mathcal{A}$ are real,

(b) all the eigenvalues of $\mathcal{A}$ are nonzero if $\det(\mathcal{A}) \neq 0$,

(c) $\mathcal{A}$ is positive definite if all its eigenvalues are positive,

(d) $\mathcal{A}$ is negative definite if all its eigenvalues are negative,

(e) $\mathcal{A}$ is positive semidefinite if all its eigenvalues are nonnegative,

(f) $\mathcal{A}$ is negative semidefinite if all its eigenvalues are nonpositive,

(g) $\mathcal{A}$ is indefinite if it has at least one positive eigenvalue and at least one negative eigenvalue.

THEOREM 8

Let $\mathcal{A} = (a_{ij})_{i,j=1}^{n}$ be a real symmetric matrix and consider the determinants

$$D_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & a_{22} & \cdots & a_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ii} \end{vmatrix} \qquad \text{for } 1 \le i \le n.$$

Thus, $D_1 = a_{11}$, $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{12}^2$, etc.

(a) If $D_i > 0$ for $1 \le i \le n$, then $\mathcal{A}$ is positive definite.

(b) If $D_i > 0$ for even numbers i in $\{1, 2, \dots, n\}$, and $D_i < 0$ for odd numbers i in $\{1, 2, \dots, n\}$, then $\mathcal{A}$ is negative definite.

(c) If $\det(\mathcal{A}) = D_n \neq 0$ but neither of the above conditions hold, then $Q(\mathbf{x})$ is indefinite.

(d) If $\det(\mathcal{A}) = 0$, then $\mathcal{A}$ is not positive or negative definite and may be semidefinite or indefinite.

EXAMPLE 9 For the matrix $\mathcal{A}$ of Example 8, we have

$$D_1 = 3 > 0, \quad D_2 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 5 > 0, \quad D_3 = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 10 > 0,$$

which reconfirms that the quadratic form of that exercise is positive definite.

EXERCISES 10.7

Evaluate the matrix products in Exercises 1–4.

1. $\begin{pmatrix} 3 & 0 & -2 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & -2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

3. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}$

4. $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

5. Evaluate $\mathcal{A}\mathcal{A}^T$ and $\mathcal{A}^2 = \mathcal{A}\mathcal{A}$, where

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

6. Evaluate $\mathbf{x}\mathbf{x}^T$, $\mathbf{x}^T\mathbf{x}$, and $\mathbf{x}^T\mathcal{A}\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} a & p & q \\ p & b & r \\ q & r & c \end{pmatrix}.$$

Evaluate the determinants in Exercises 7–8.

7. $\begin{vmatrix} 2 & 3 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 1 & 0 & -1 & 1 \\ -2 & 0 & 0 & 1 \end{vmatrix}$ **8.** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 4 \\ 3 & -3 & 2 & -2 \end{vmatrix}$

9. Show that if $\mathcal{A} = (a_{ij})$ is an $n \times n$ matrix for which $a_{ij} = 0$ whenever $i > j$, then $\det(\mathcal{A}) = \prod_{k=1}^{n} a_{kk}$, the product of the elements on the main diagonal of $\mathcal{A}$.

10. Show that $\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = y - x$, and

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (y - x)(z - x)(z - y).$$

Try to generalize this result to the $n \times n$ case.

11. Verify the associative law $(\mathcal{AB})\mathcal{C} = \mathcal{A}(\mathcal{BC})$ by direct calculation for three arbitrary 2×2 matrices.

12. Show that $\det(\mathcal{A}^T) = \det(\mathcal{A})$ for $n \times n$ matrices by induction on n. Start with the 2×2 case.

13. Verify by direct calculation that $\det(\mathcal{AB}) = \det(\mathcal{A})\det(\mathcal{B})$ holds for two arbitrary 2×2 matrices.

14. Let $\mathcal{A}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Show that $(\mathcal{A}_\theta)^T = (\mathcal{A}_\theta)^{-1} = \mathcal{A}_{-\theta}$.

15. Verify by using matrix multiplication that the inverse of the matrix $\mathcal{A}$ in the remark following Example 5 is as specified there.

16. For what values of the variables x and y is the matrix $\mathcal{B} = \begin{pmatrix} x & y \\ x^2 & y^2 \end{pmatrix}$ invertible, and what is its inverse?

Find the inverses of the matrices in Exercises 17–18.

17. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ **18.** $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix}$

19. Use your result from Exercise18 to solve the linear system

$$\begin{cases} x - z = -2 \\ -x + y = 1 \\ 2x + y + 3z = 13. \end{cases}$$

20. Solve the system of Exercise 19 by using Cramer's Rule.

21. Solve the system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 4 \\ x_1 + x_2 - x_3 - x_4 = 6 \\ x_1 - x_2 - x_3 - x_4 = 2. \end{cases}$

22. Verify Theorem 5 for the special case where **F** and **G** are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

In Exercises 23–28, classify the given symmetric matrices as positive or negative definite, positive or negative semidefinite, or indefinite.

23. $\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$ **24.** $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

25. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ **26.** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

27. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ **28.** $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

10.8 Using Maple for Vector and Matrix Calculations

The use of a computer algebra system can free us from much of the tedious calculation needed to do calculus. This is especially true of calculations in multivariable and vector calculus, where the calculations can quickly become unmanageable as the number of variables increases. This author's colleague, Dr. Robert Israel, has written an excellent book, *Calculus, the Maple Way*, to show how Maple can be used effectively for doing calculus involving both single-variable and multivariable functions.

In this book we will occasionally call on the power of Maple to carry out calculations involving functions of several variables and vector-valued functions of one or more variables. This section illustrates some of the most basic techniques for calculating with vectors and matrices. The examples here were calculated using Maple 10, but Maple 6 or later should give similar output.

Most of Maple's capability to deal with vectors and matrices is not in its kernel but is written into a package of procedures called **LinearAlgebra**. Therefore, it is customary to load this package at the beginning of a session where it will be needed:

```
> with(LinearAlgebra):
```

One usually completes a Maple command with a semicolon rather than a colon. You can use a colon to suppress output. Had we used a semicolon to complete the command