32. **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that **u**, **v**, and **r** are position vectors of points U, V, and P, respectively, that **u** is not parallel to **v**, and that P lies in the plane containing the origin, U, and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both **v** and **r** as sums of vectors parallel and perpendicular to **u** as suggested in Exercise 31.

33. Given constants r, s, and t, with $r \neq 0$ and $s \neq 0$, and given a vector **a** satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors **x** and **y**.

Hanging cables

34. **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is δgx rather than δgs, show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

10.3 The Cross Product in 3-Space

There is defined, *in 3-space only*, another kind of product of two vectors called a *cross product* or *vector product*, and denoted $\mathbf{u}\times\mathbf{v}$.

DEFINITION 5

For any vectors **u** and **v** in $\mathbb{R}^3$, the **cross product** $\mathbf{u}\times\mathbf{v}$ is the unique vector satisfying the following three conditions:

(i) $(\mathbf{u}\times\mathbf{v}) \bullet \mathbf{u} = 0$ and $(\mathbf{u}\times\mathbf{v}) \bullet \mathbf{v} = 0$,

(ii) $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$, where θ is the angle between **u** and **v**, and

(iii) **u**, **v**, and $\mathbf{u}\times\mathbf{v}$ form a right-handed triad.

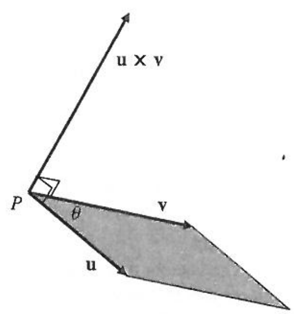

Figure 10.22 $\mathbf{u}\times\mathbf{v}$ is perpendicular to both **u** and **v** and has length equal to the area of the shaded parallelogram

If **u** and **v** are parallel, condition (ii) says that $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, the zero vector. Otherwise, through any point in $\mathbb{R}^3$ there is a unique straight line that is perpendicular to both **u** and **v**. Condition (i) says that $\mathbf{u}\times\mathbf{v}$ is parallel to this line. Condition (iii) determines which of the two directions along this line is the direction of $\mathbf{u}\times\mathbf{v}$; a right-handed screw advances in the direction of $\mathbf{u}\times\mathbf{v}$ if rotated in the direction from **u** toward **v**. (This is equivalent to saying that the thumb, forefinger, and middle finger of the right hand can be made to point in the directions of **u**, **v**, and $\mathbf{u}\times\mathbf{v}$, respectively.)

If **u** and **v** have their tails at the point P, then $\mathbf{u}\times\mathbf{v}$ is normal (i.e., perpendicular) to the plane through P in which **u** and **v** lie and, by condition (ii), $\mathbf{u}\times\mathbf{v}$ has length equal to the area of the parallelogram spanned by **u** and **v**. (See Figure 10.22.) These properties make the cross product very useful for the description of tangent planes and normal lines to surfaces in $\mathbb{R}^3$.

The definition of cross product given above does not involve any coordinate system and therefore does not directly show the components of the cross product with respect to the standard basis. These components are provided by the following theorem:

THEOREM 2

Components of the cross product

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

PROOF First, we observe that the vector

$$\mathbf{w} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

is perpendicular to both **u** and **v** since

$$\mathbf{u}\bullet\mathbf{w} = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0,$$

and similarly $\mathbf{v}\bullet\mathbf{w} = 0$. Thus, $\mathbf{u}\times\mathbf{v}$ is parallel to **w**. Next, we show that **w** and $\mathbf{u}\times\mathbf{v}$ have the same length. In fact,

$$\begin{aligned}|\mathbf{w}|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2\\ &= u_2^2v_3^2 + u_3^2v_2^2 - 2u_2v_3u_3v_2 + u_3^2v_1^2 + u_1^2v_3^2\\ &\quad - 2u_3v_1u_1v_3 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_1v_2u_2v_1,\end{aligned}$$

while

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta\\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta)\\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\bullet\mathbf{v})^2\\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2\\ &= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2\\ &\quad - u_1^2v_1^2 - u_2^2v_2^2 - u_3^2v_3^2 - 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\\ &= |\mathbf{w}|^2.\end{aligned}$$

Since **w** is parallel to, and has the same length as, $\mathbf{u}\times\mathbf{v}$, we must have either $\mathbf{u}\times\mathbf{v} = \mathbf{w}$ or $\mathbf{u}\times\mathbf{v} = -\mathbf{w}$. It remains to be shown that the first of these is the correct choice. To see this, suppose that the triad of vectors **u**, **v**, and **w** is rigidly rotated in 3-space so that **u** points in the direction of the positive x-axis and **v** lies in the upper half of the xy-plane. Then $\mathbf{u} = u_1\mathbf{i}$, and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, where $u_1 > 0$ and $v_2 > 0$. By the "right-hand rule" $\mathbf{u}\times\mathbf{v}$ must point in the direction of the positive z-axis. But $\mathbf{w} = u_1v_2\mathbf{k}$ does point in that direction, so $\mathbf{u}\times\mathbf{v} = \mathbf{w}$, as asserted.

The formula for the cross product in terms of components may seem awkward and asymmetric. As we shall see, however, it can be written more easily in terms of a determinant. We introduce determinants later in this section.

EXAMPLE 1 **(Calculating cross products)**

(a)

$$\begin{array}{lll}\mathbf{i}\times\mathbf{i} = \mathbf{0}, & \mathbf{i}\times\mathbf{j} = \mathbf{k}, & \mathbf{j}\times\mathbf{i} = -\mathbf{k},\\ \mathbf{j}\times\mathbf{j} = \mathbf{0}, & \mathbf{j}\times\mathbf{k} = \mathbf{i}, & \mathbf{k}\times\mathbf{j} = -\mathbf{i},\\ \mathbf{k}\times\mathbf{k} = \mathbf{0}, & \mathbf{k}\times\mathbf{i} = \mathbf{j}, & \mathbf{i}\times\mathbf{k} = -\mathbf{j}.\end{array}$$

(b) $(2\mathbf{i}+\mathbf{j}-3\mathbf{k})\times(-2\mathbf{j}+5\mathbf{k})$

$$\begin{aligned}&= \big((1)(5) - (-2)(-3)\big)\mathbf{i} + \big((-3)(0) - (2)(5)\big)\mathbf{j} + \big((2)(-2) - (1)(0)\big)\mathbf{k}\\ &= -\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

The cross product has some but not all of the properties we usually ascribe to products. We summarize its algebraic properties as follows:

Properties of the cross product

If **u**, **v**, and **w** are any vectors in $\mathbb{R}^3$, and t is a real number (a scalar), then

(i) $\mathbf{u}\times\mathbf{u} = \mathbf{0}$,

(ii) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$, (The cross product is **anticommutative**.)

(iii) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$,

(iv) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$,

(v) $(t\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(t\mathbf{v}) = t(\mathbf{u}\times\mathbf{v})$,

(vi) $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\bullet(\mathbf{u}\times\mathbf{v}) = 0$.

These identities are all easily verified using the components or the definition of the cross product or by using properties of determinants discussed below. They are left as exercises for the reader. Note the absence of an associative law. The cross product is not associative. (See Exercise 21 at the end of this section.) In general,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}.$$

Determinants

In order to simplify certain formulas such as the component representation of the cross product, we introduce 2×2 and 3×3 **determinants**. General $n \times n$ determinants are normally studied in courses on linear algebra; we will encounter them in Section 10.6. In this section we will outline enough of the properties of determinants to enable us to use them as shorthand in some otherwise complicated formulas.

A determinant is an expression that involves the elements of a square array (matrix) of numbers. The determinant of the 2×2 array of numbers

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

is denoted by enclosing the array between vertical bars, and its value is the number $ad - bc$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Figure 10.23 Upward and downward diagonals

This is the product of elements in the *downward diagonal* of the array minus the product of elements in the *upward diagonal* as shown in Figure 10.23. For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2.$$

Similarly, the determinant of a 3×3 array of numbers is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

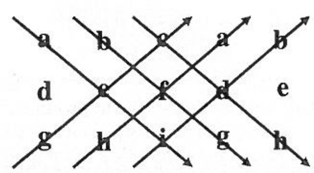

Figure 10.24 WARNING: This method does not work for 4×4 or higher-order determinants!

Observe that each of the six products in the value of the determinant involves exactly one element from each row and exactly one from each column of the array. As such, each term is the product of elements in a *diagonal* of an *extended* array obtained by repeating the first two columns of the array to the right of the third column, as shown in Figure 10.24. The value of the determinant is the sum of products corresponding to the three complete *downward* diagonals minus the sum corresponding to the three *upward* diagonals. With practice you will be able to form these diagonal products without having to write the extended array.

If we group the terms in the expansion of the determinant to factor out the elements of the first row, we obtain

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}. \end{aligned}$$

The 2×2 determinants appearing here (called *minors* of the given 3×3 determinant) are obtained by deleting the row and column containing the corresponding element from the original 3×3 determinant. This process is called *expanding* the 3×3 determinant *in minors* about the first row.

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The pattern of + and − signs used with the terms of an expansion in minors of a 3 × 3 determinant is given by

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Such expansions in minors can be carried out about any row or column. Note that if $i + j$ is an *odd* number, a minus sign appears in a term obtained by multiplying the element in the ith row and jth column and its corresponding minor obtained by deleting that row and column. For example, we can expand the above determinant in minors about the second column as follows:

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ &= -bdi + bfg + eai - ecg - haf + hcd.\end{aligned}$$

(Of course, this is the same value as the one obtained previously.)

EXAMPLE 2

$$\begin{aligned}\begin{vmatrix} 1 & 4 & -2 \\ -3 & 1 & 0 \\ 2 & 2 & -3 \end{vmatrix} &= 3\begin{vmatrix} 4 & -2 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \\ &= 3(-8) + 1 = -23.\end{aligned}$$

We expanded about the second row; the third column would also have been a good choice. (Why?)

Any row (or column) of a determinant may be regarded as the components of a vector. Then the determinant is a *linear function* of that vector. For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ sx + tl & sy + tm & sz + tn \end{vmatrix} = s\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} + t\begin{vmatrix} a & b & c \\ d & e & f \\ l & m & n \end{vmatrix}$$

because the determinant is a linear function of its third row. This and other properties of determinants follow directly from the definition. Some other properties are summarized below. These are stated for rows and for 3 × 3 determinants, but similar statements can be made for columns and for determinants of any order.

Properties of determinants

(i) If two rows of a determinant are interchanged, then the determinant changes sign:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

(ii) If two rows of a determinant are equal, the determinant has value 0:

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = 0.$$

(iii) If a multiple of one row of a determinant is added to another row, the value of the determinant remains unchanged:

$$\begin{vmatrix} a & b & c \\ d + ta & e + tb & f + tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

The Cross Product as a Determinant

The elements of a determinant are usually numbers because they have to be multiplied to get the value of the determinant. However, it is possible to use vectors as the elements of *one row* (or column) of a determinant. When expanding in minors about that row (or column), the minor for each vector element is a number that determines the scalar multiple of the vector. The formula for the cross product of

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

presented in Theorem 2 can be expressed symbolically as a determinant with the standard basis vectors as the elements of the first row:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

The formula for the cross product given in that theorem is just the expansion of this determinant in minors about the first row.

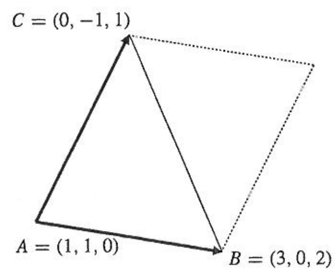

Figure 10.25

EXAMPLE 3 Find the area of the triangle with vertices at the three points $A = (1, 1, 0)$, $B = (3, 0, 2)$, and $C = (0, -1, 1)$.

Solution Two sides of the triangle (Figure 10.25) are given by the vectors:

$$\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \overrightarrow{AC} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The area of the triangle is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. By the definition of cross product, the area of the triangle must therefore be

$$\begin{aligned} \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| &= \frac{1}{2}\left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & -2 & 1 \end{vmatrix} \right| \\ &= \frac{1}{2}|3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}| = \frac{1}{2}\sqrt{9 + 16 + 25} = \frac{5}{2}\sqrt{2} \text{ square units.} \end{aligned}$$

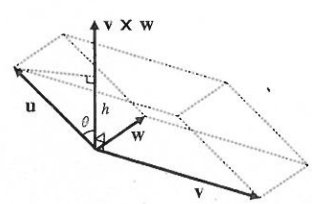

Figure 10.26

A **parallelepiped** is the three-dimensional analogue of a parallelogram. It is a solid with three pairs of parallel planar faces. Each face is in the shape of a parallelogram. A rectangular brick is a special case of a parallelepiped in which nonparallel faces intersect at right angles. We say that a parallelepiped is **spanned** by three vectors coinciding with three of its edges that meet at one vertex. (See Figure 10.26.)

EXAMPLE 4 Find the volume of the parallelepiped spanned by the vectors **u**, **v**, and **w**.

Solution The volume of the parallelepiped is equal to the area of one of its faces, say, the face spanned by **v** and **w**, multiplied by the height of the parallelepiped measured in a direction perpendicular to that face. The area of the face is $|\mathbf{v} \times \mathbf{w}|$. Since $\mathbf{v} \times \mathbf{w}$ is perpendicular to the face, the height h of the parallelepiped will be the absolute value of the scalar projection of **u** along $\mathbf{v} \times \mathbf{w}$. If θ is the angle between **u** and $\mathbf{v} \times \mathbf{w}$, then the volume of the parallelepiped is given by

$$\text{Volume} = |\mathbf{u}||\mathbf{v} \times \mathbf{w}||\cos\theta| = |\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| \text{ cubic units.}$$

DEFINITION 6

The quantity $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ is called the **scalar triple product** of the vectors **u**, **v**, and **w**.

The scalar triple product is easily expressed in terms of a determinant. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, and similar representations hold for $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of this determinant.

Using the properties of the determinant, it is easily verified that

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}).$$

(See Exercise 18 below.) Note that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ remain in the same *cyclic order* in these three expressions. Reversing the order would introduce a factor -1:

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = -\mathbf{u} \bullet (\mathbf{w} \times \mathbf{v}).$$

Three vectors in 3-space are said to be **coplanar** if the parallelepiped they span has zero volume; if their tails coincide, three such vectors must lie in the same plane.

$$\mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \text{ are coplanar} \iff \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 0 \iff \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Three vectors are certainly coplanar if any of them is $\mathbf{0}$, or if any pair of them is parallel. If neither of these degenerate conditions apply, they are only coplanar if any one of them can be expressed as a linear combination of the other two. (See Exercise 20 below.)

Applications of Cross Products

Cross products are of considerable importance in mechanics and electromagnetic theory, as well as in the study of motion in general. For example:

(a) The linear velocity $\mathbf{v}$ of a particle located at position $\mathbf{r}$ in a body rotating with angular velocity $\mathbf{\Omega}$ about the origin is given by $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$. (See Section 11.2 for more details.)

(b) The angular momentum of a planet of mass m moving with velocity $\mathbf{v}$ in its orbit around the sun is given by $\mathbf{h} = \mathbf{r} \times m\mathbf{v}$, where $\mathbf{r}$ is the position vector of the planet relative to the sun as origin. (See Section 11.6.)

(c) If a particle of electric charge q is travelling with velocity $\mathbf{v}$ through a magnetic field whose strength and direction are given by vector $\mathbf{B}$, then the force that the field exerts on the particle is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. The electron beam in a television tube is controlled by magnetic fields using this principle.

(d) The torque $\mathbf{T}$ of a force $\mathbf{F}$ applied at the point P with position vector $\mathbf{r}$ about another point P_0 with position vector $\mathbf{r}_0$ is defined to be

$$\mathbf{T} = \overrightarrow{P_0P} \times \mathbf{F} = (\mathbf{r} - \mathbf{r}_0) \times \mathbf{F}.$$

This torque measures the effectiveness of the force $\mathbf{F}$ in causing rotation about P_0. The direction of $\mathbf{T}$ is along the axis through P_0 about which $\mathbf{F}$ acts to rotate P.

EXAMPLE 5 An automobile wheel has centre at the origin and axle along the y-axis. One of the retaining nuts holding the wheel is at position $P_0 = (0, 0, 10)$. (Distances are measured in centimetres.) A bent tire wrench with arm 25 cm long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction, as shown in Figure 10.27. If a horizontal force $\mathbf{F} = 500\mathbf{i}$ newtons (N) is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?

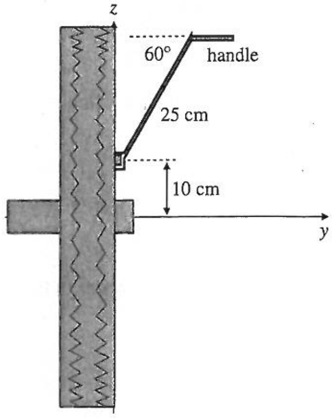

Figure 10.27 The force on the handle is 500 N in a direction directly toward you

Solution The nut is at position $\mathbf{r}_0 = 10\mathbf{k}$, and the handle of the wrench is at position

$$\mathbf{r} = 25\cos 60°\mathbf{j} + (10 + 25\sin 60°)\mathbf{k} \approx 12.5\mathbf{j} + 31.65\mathbf{k}.$$

The torque of the force $\mathbf{F}$ on the nut is

$$\begin{aligned}\mathbf{T} &= (\mathbf{r} - \mathbf{r}_0) \times \mathbf{F} \\ &\approx (12.5\mathbf{j} + 21.65\mathbf{k}) \times 500\mathbf{i} \approx 10{,}825\mathbf{j} - 6{,}250\mathbf{k},\end{aligned}$$

which is at right angles to $\mathbf{F}$ and to the arm of the wrench. Only the horizontal component of this torque is effective in turning the nut. This component is 10,825 N·cm or 108.25 N·m in magnitude. For the effective torque on the wheel itself, we have to replace $\mathbf{r}_0$ by $\mathbf{0}$, the position of the centre of the wheel. In this case the horizontal torque is

$$31.65\mathbf{k} \times 500\mathbf{i} \approx 15{,}825\mathbf{j},$$

that is, about 158.25 N·m.

EXERCISES 10.3

1. Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.
2. Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$.
3. Find the area of the triangle with vertices $(1, 2, 0)$, $(1, 0, 2)$, and $(0, 3, 1)$.
4. Find a unit vector perpendicular to the plane containing the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. What is the area of the triangle with these vertices?
5. Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$.
6. Find a unit vector with positive $\mathbf{k}$ component that is perpendicular to both $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Verify the identities in Exercises 7–11, either by using the definition of cross product or the properties of determinants.

7. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
8. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
9. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
10. $(t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$
11. $\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$
12. Obtain the addition formula

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

by examining the cross product of the two unit vectors $\mathbf{u} = \cos\beta\mathbf{i} + \sin\beta\mathbf{j}$ and $\mathbf{v} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$. Assume $0 \le \alpha - \beta \le \pi$. *Hint:* Regard $\mathbf{u}$ and $\mathbf{v}$ as position vectors. What is the area of the parallelogram they span?

13. If $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$.
14. **(Volume of a tetrahedron)** A **tetrahedron** is a pyramid with a triangular base and three other triangular faces. It has four vertices and six edges. Like any pyramid or cone, its volume is equal to $\frac{1}{3}Ah$, where A is the area of the base and h is the height measured perpendicular to the base. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors coinciding with the three edges of a tetrahedron that meet at one vertex, show that the tetrahedron has volume given by

$$\text{Volume} = \frac{1}{6}\,|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = \frac{1}{6}\left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|.$$

Thus, the volume of a tetrahedron spanned by three vectors is one-sixth of the volume of the parallelepiped spanned by the same vectors.

15. Find the volume of the tetrahedron with vertices $(1, 0, 0)$, $(1, 2, 0)$, $(2, 2, 2)$, and $(0, 3, 2)$.
16. Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

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17. For what value of k do the four points $(1, 1, -1)$, $(0, 3, -2)$, $(-2, 1, 0)$, and $(k, 0, 2)$ all lie in a plane?

18. **(The scalar triple product)** Verify the identities

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}).$$

19. If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) \neq 0$ and $\mathbf{x}$ is an arbitrary 3-vector, find the numbers λ, μ, and ν such that

$$\mathbf{x} = \lambda \mathbf{u} + \mu \mathbf{v} + \nu \mathbf{w}.$$

20. If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 0$ but $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$, show that there are constants λ and μ such that

$$\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w}.$$

Hint: Use the result of Exercise 19 with **u** in place of **x** and $\mathbf{v} \times \mathbf{w}$ in place of **u**.

21. Calculate $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = \mathbf{j} - \mathbf{k}$. Why would you not expect these to be equal?

22. Does the notation $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ make sense? Why? How about the notation $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$?

23. **(The vector triple product)** The product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called a **vector triple product**. Since it is perpendicular to $\mathbf{v} \times \mathbf{w}$, it must lie in the plane of **v** and **w**. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}.$$

Hint: This can be done by direct calculation of the components of both sides of the equation, but the job is much easier if you choose coordinate axes so that **v** lies along the x-axis and **w** lies in the xy-plane.

24. If **u**, **v**, and **w** are mutually perpendicular vectors, show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$. What is $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ in this case?

25. Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

26. Find all vectors **x** that satisfy the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{x} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

27. Show that the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{x} = \mathbf{i} + 5\mathbf{j}$$

has no solutions for the unknown vector **x**.

28. What condition must be satisfied by the nonzero vectors **a** and **b** to guarantee that the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ has a solution for **x**? Is the solution unique?

10.4 Planes and Lines

A single equation in the three variables, x, y, and z, constitutes a single constraint on the freedom of the point $P = (x, y, z)$ to lie anywhere in 3-space. Such a constraint usually results in the loss of exactly one *degree of freedom* and so forces P to lie on a two-dimensional surface. For example, the equation

$$x^2 + y^2 + z^2 = 4$$

states that the point (x, y, z) is at distance 2 from the origin. All points satisfying this condition lie on a **sphere** (i.e., the surface of a ball) of radius 2 centred at the origin. The equation above therefore represents that sphere, and the sphere is the graph of the equation. In this section we will investigate the graphs of linear equations in three variables.

Planes in 3-Space

Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ with position vector

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}.$$

If $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is any given *nonzero* vector, then there exists exactly one **plane** (flat surface) passing through P_0 and perpendicular to **n**. We say that **n** is a **normal vector** to the plane. The plane is the set of all points P for which $\overrightarrow{P_0P}$ is perpendicular to **n**. (See Figure 10.28.)

If $P = (x, y, z)$ has position vector **r**, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. This vector is perpendicular to **n** if and only if $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r}_0) = 0$. This is the equation of the plane in vector form. We can rewrite it in terms of coordinates to obtain the corresponding scalar equation.