17. $(x-1)^2+(y+2)^2+(z-3)^2=4$

18. $x^2+y^2+z^2=2z$

19. $y^2+z^2\le 4$

20. $x^2+z^2=4$

21. $z=y^2$

22. $z\ge\sqrt{x^2+y^2}$

23. $x+2y+3z=6$

In Exercises 24–32, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

24. $\begin{cases}x=1\\y=2\end{cases}$

25. $\begin{cases}x=1\\y=z\end{cases}$

26. $\begin{cases}x^2+y^2+z^2=4\\z=1\end{cases}$

27. $\begin{cases}x^2+y^2+z^2=4\\x^2+y^2+z^2=4x\end{cases}$

28. $\begin{cases}x^2+y^2+z^2=4\\x^2+z^2=1\end{cases}$

29. $\begin{cases}x^2+y^2=1\\z=x\end{cases}$

30. $\begin{cases}y\ge x\\z\le y\end{cases}$

31. $\begin{cases}x^2+y^2\le 1\\z\ge y\end{cases}$

32. $\begin{cases}x^2+y^2+z^2\le 1\\\sqrt{x^2+y^2}\le z\end{cases}$

In Exercises 33–36, specify the boundary and the interior of the plane sets S whose points (x, y) satisfy the given conditions. Is S open, closed, or neither?

33. $0<x^2+y^2<1$

34. $x\ge 0,\quad y<0$

35. $x+y=1$

36. $|x|+|y|\le 1$

In Exercises 37–40, specify the boundary and the interior of the sets S in 3-space whose points (x, y, z) satisfy the given conditions. Is S open, closed, or neither?

37. $1\le x^2+y^2+z^2\le 4$

38. $x\ge 0,\quad y>1,\quad z<2$

39. $(x-z)^2+(y-z)^2=0$

40. $x^2+y^2<1,\ y+z>2$

10.2 Vectors

A **vector** is a quantity that involves both **magnitude** (size or length) and **direction**. For instance, the *velocity* of a moving object involves its speed and direction of motion, so is a vector. Such quantities are represented geometrically by arrows (directed line segments) and are often actually identified with these arrows. For instance, the vector $\overrightarrow{AB}$ is an arrow with tail at the point A and head at the point B. In print, such a vector is usually denoted by a single letter in boldface type,

$$\mathbf{v}=\overrightarrow{AB}.$$

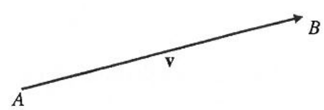

Figure 10.11 The vector $\mathbf{v}=\overrightarrow{AB}$

(See Figure 10.11.) In handwriting, an arrow over a letter ($\vec{v}=\overrightarrow{AB}$) can be used to denote a vector. The *magnitude* of the vector $\mathbf{v}$ is the length of the arrow and is denoted $|\mathbf{v}|$ or $|\overrightarrow{AB}|$.

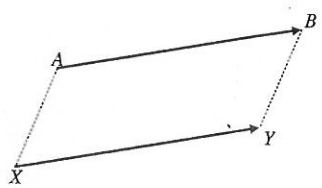

Figure 10.12 $\overrightarrow{AB}=\overrightarrow{XY}$

While vectors have magnitude and direction, they do not generally have *position*; that is, they are not regarded as being in a particular place. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are considered *equal* if they have *the same length and the same direction*, even if their representative arrows do not coincide. The arrows must be parallel, have the same length, and point in the same direction. In Figure 10.12, for example, if $ABYX$ is a parallelogram, then $\overrightarrow{AB}=\overrightarrow{XY}$.

For the moment, we consider plane vectors, that is, vectors whose representative arrows lie in a plane. If we introduce a Cartesian coordinate system into the plane, we can talk about the x and y components of any vector. If $A=(a,b)$ and $P=(p,q)$, as shown in Figure 10.13, then the x and y components of $\overrightarrow{AP}$ are, respectively, $p-a$ and $q-b$. Note that if O is the origin and X is the point $(p-a, q-b)$, then

$$|\overrightarrow{AP}|=\sqrt{(p-a)^2+(q-b)^2}=|\overrightarrow{OX}|$$

$$\text{slope of }\overrightarrow{AP}=\frac{q-b}{p-a}=\text{slope of }\overrightarrow{OX}.$$

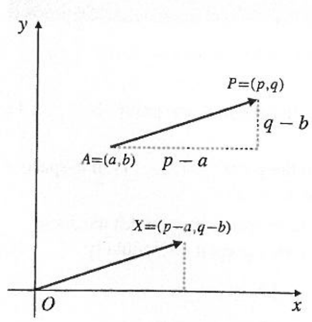

Figure 10.13 Components of a vector

Hence $\overrightarrow{AP}=\overrightarrow{OX}$. In general, two vectors are equal if and only if they have the same x components and y components.

There are two important algebraic operations defined for vectors: addition and scalar multiplication.

DEFINITION 1

Vector addition

Given two vectors **u** and **v**, their **sum** $\mathbf{u} + \mathbf{v}$ is defined as follows. If an arrow representing **v** is placed with its tail at the head of an arrow representing **u**, then an arrow from the tail of **u** to the head of **v** represents $\mathbf{u} + \mathbf{v}$. Equivalently, if **u** and **v** have tails at the same point, then $\mathbf{u} + \mathbf{v}$ is represented by an arrow with its tail at that point and its head at the opposite vertex of the parallelogram spanned by **u** and **v**. This is shown in Figure 10.14(a).

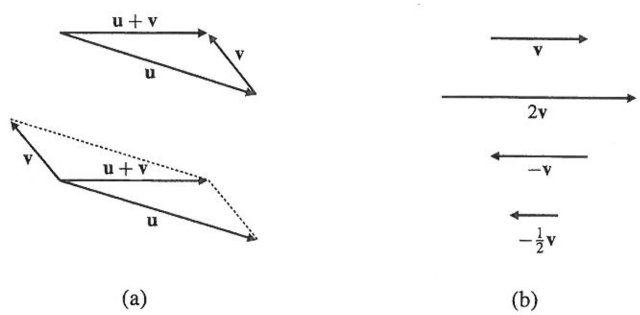

Figure 10.14
(a) Vector addition
(b) Scalar multiplication

DEFINITION 2

Scalar multiplication

If **v** is a vector and t is a real number (also called a **scalar**), then the **scalar multiple** $t\mathbf{v}$ is a vector with magnitude $|t|$ times that of **v** and direction the same as **v** if $t > 0$, or opposite to that of **v** if $t < 0$. See Figure 10.14(b). If $t = 0$, then $t\mathbf{v}$ has zero length and therefore no particular direction. It is the **zero vector**, denoted **0**.

Suppose that **u** has components a and b and that **v** has components x and y. Then the components of $\mathbf{u} + \mathbf{v}$ are $a + x$ and $b + y$, and those of $t\mathbf{v}$ are tx and ty. See Figure 10.15.

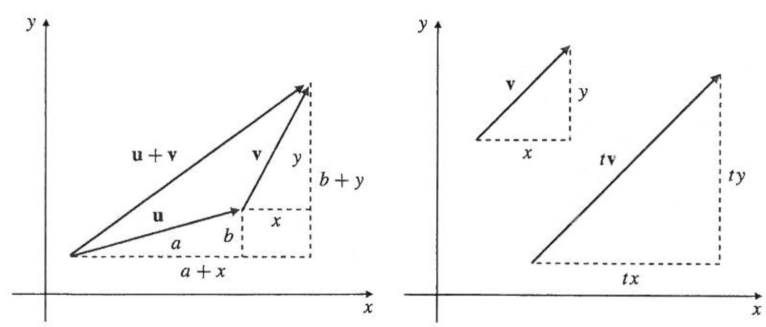

Figure 10.15 The components of a sum of vectors or a scalar multiple of a vector is the same sum or multiple of the corresponding components of the vectors

In $\mathbb{R}^2$ we single out two particular vectors for special attention. They are

(i) the vector **i** from the origin to the point $(1, 0)$, and

(ii) the vector **j** from the origin to the point $(0, 1)$.

Thus, **i** has components 1 and 0, and **j** has components 0 and 1. These vectors are called the **standard basis vectors** in the plane. The vector **r** from the origin to the point (x, y) has components x and y and can be expressed in the form

$$\mathbf{r} = \langle x, y \rangle = x\mathbf{i} + y\mathbf{j}.$$

In the first form we specify the vector by listing its components between angle brackets; in the second we write $\mathbf{r}$ as a **linear combination** of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$. (See Figure 10.16.) The vector $\mathbf{r}$ is called the **position vector** of the point (x, y). A position vector has its tail at the origin and its head at the point whose position it is specifying. The length of $\mathbf{r}$ is $|\mathbf{r}| = \sqrt{x^2 + y^2}$.

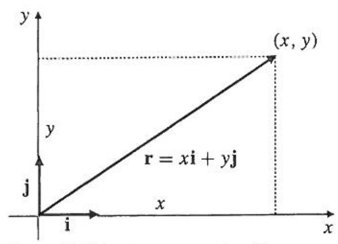

Figure 10.16 Any vector is a linear combination of the basis vectors

More generally, the vector $\overrightarrow{AP}$ from $A = (a, b)$ to $P = (p, q)$ in Figure 10.13 can also be written as a list of components or as a linear combination of the standard basis vectors:

$$\overrightarrow{AP} = \langle p - a, q - b \rangle = (p - a)\mathbf{i} + (q - b)\mathbf{j}.$$

Sums and scalar multiples of vectors are easily expressed in terms of components. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, and if t is a scalar (i.e., a real number), then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j},$$
$$t\mathbf{u} = (tu_1)\mathbf{i} + (tu_2)\mathbf{j}.$$

The zero vector is $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$. It has length zero and no specific direction. For any vector $\mathbf{u}$ we have $0\mathbf{u} = \mathbf{0}$. A **unit vector** is a vector of length 1. The standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors. Given any nonzero vector $\mathbf{v}$, we can form a unit vector $\hat{\mathbf{v}}$ in the same direction as $\mathbf{v}$ by multiplying $\mathbf{v}$ by the reciprocal of its length (a scalar):

$$\hat{\mathbf{v}} = \left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v}.$$

EXAMPLE 1 If $A = (2, -1)$, $B = (-1, 3)$, and $C = (0, 1)$, express each of the following vectors as a linear combination of the standard basis vectors:

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{AB} + \overrightarrow{BC}$ (e) $2\overrightarrow{AC} - 3\overrightarrow{CB}$

(f) a unit vector in the direction of $\overrightarrow{AB}$.

Solution

(a) $\overrightarrow{AB} = (-1 - 2)\mathbf{i} + (3 - (-1))\mathbf{j} = -3\mathbf{i} + 4\mathbf{j}$

(b) $\overrightarrow{BC} = (0 - (-1))\mathbf{i} + (1 - 3)\mathbf{j} = \mathbf{i} - 2\mathbf{j}$

(c) $\overrightarrow{AC} = (0 - 2)\mathbf{i} + (1 - (-1))\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$

(d) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = -2\mathbf{i} + 2\mathbf{j}$

(e) $2\overrightarrow{AC} - 3\overrightarrow{CB} = 2(-2\mathbf{i} + 2\mathbf{j}) - 3(-\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} - 2\mathbf{j}$

(f) A unit vector in the direction of $\overrightarrow{AB}$ is $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = -\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$.

Implicit in the above example is the fact that the operations of addition and scalar multiplication obey appropriate algebraic rules, such as

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}),$$
$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v},$$
$$t(\mathbf{u} + \mathbf{v}) = t\mathbf{u} + t\mathbf{v}.$$

Vectors in 3-Space

The algebra and geometry of vectors described here extends to spaces of any number of dimensions; we can still think of vectors as represented by arrows, and sums and scalar multiples are formed just as for plane vectors.

Given a Cartesian coordinate system in 3-space, we define three **standard basis vectors, i, j,** and **k**, represented by arrows from the origin to the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. (See Figure 10.17.) Any vector in 3-space can be written as a *linear combination* of these basis vectors; for instance, the position vector of the point (x, y, z) is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We say that **r** has **components** x, y, and z. The length of **r** is

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

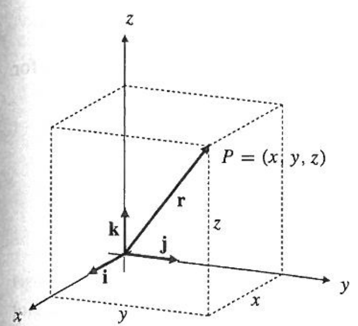

Figure 10.17 The standard basis vectors **i**, **j**, and **k**

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in 3-space, then the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ from P_1 to P_2 has components $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ and is therefore represented in terms of the standard basis vectors by

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

EXAMPLE 2 If $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{u} - 2\mathbf{v}$, $|\mathbf{u}|$, $|\mathbf{v}|$, and a unit vector $\hat{\mathbf{u}}$ in the direction of **u**.

Solution

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (2+3)\mathbf{i} + (1-2)\mathbf{j} + (-2-1)\mathbf{k} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}\\
\mathbf{u} - \mathbf{v} &= (2-3)\mathbf{i} + (1+2)\mathbf{j} + (-2+1)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}\\
3\mathbf{u} - 2\mathbf{v} &= (6-6)\mathbf{i} + (3+4)\mathbf{j} + (-6+2)\mathbf{k} = 7\mathbf{j} - 4\mathbf{k}\\
|\mathbf{u}| &= \sqrt{4+1+4} = 3, \qquad |\mathbf{v}| = \sqrt{9+4+1} = \sqrt{14}\\
\hat{\mathbf{u}} &= \left(\frac{1}{|\mathbf{u}|}\right)\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.
\end{aligned}$$

The following example illustrates the way vectors can be used to solve problems involving relative velocities. If A moves with velocity $\mathbf{v}_{A \text{ rel } B}$ relative to B, and B moves with velocity $\mathbf{v}_{B \text{ rel } C}$ relative to C, then A moves with velocity $\mathbf{v}_{A \text{ rel } C}$ relative to C, where

$$\mathbf{v}_{A \text{ rel } C} = \mathbf{v}_{A \text{ rel } B} + \mathbf{v}_{B \text{ rel } C}.$$

EXAMPLE 3 An aircraft cruises at a speed of 300 km/h in still air. If the wind is blowing from the east at 100 km/h, in what direction should the aircraft head in order to fly in a straight line from city P to city Q, 400 km north northeast of P? How long will the trip take?

Solution The problem is two-dimensional, so we use plane vectors. Let us choose our coordinate system so that the x- and y-axes point east and north, respectively. Figure 10.18 illustrates the three velocities that must be considered. The velocity of the air relative to the ground is

$$\mathbf{v}_{\text{air rel ground}} = -100\,\mathbf{i}.$$

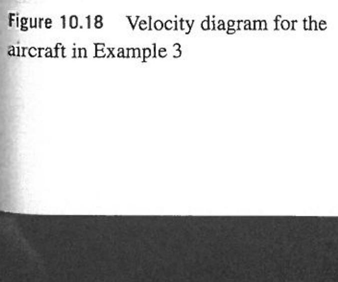

Figure 10.18 Velocity diagram for the aircraft in Example 3

If the aircraft heads in a direction making angle θ with the positive direction of the x-axis, then the velocity of the aircraft relative to the air is

$$\mathbf{v}_{\text{aircraft rel air}} = 300\cos\theta\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.$$

Thus, the velocity of the aircraft relative to the ground is

$$\begin{aligned}
\mathbf{v}_{\text{aircraft rel ground}} &= \mathbf{v}_{\text{aircraft rel air}} + \mathbf{v}_{\text{air rel ground}}\\
&= (300\cos\theta - 100)\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.
\end{aligned}$$

We want this latter velocity to be in a north-northeasterly direction, that is, in the direction making angle $3\pi/8 = 67.5°$ with the positive direction of the x-axis. Thus, we will have

$$\mathbf{v}_{\text{aircraft rel ground}} = v\left[(\cos 67.5°)\,\mathbf{i} + (\sin 67.5°)\,\mathbf{j}\right],$$

where v is the actual groundspeed of the aircraft. Comparing the two expressions for $\mathbf{v}_{\text{aircraft rel ground}}$ we obtain

$$\begin{aligned} 300\cos\theta - 100 &= v\cos 67.5° \\ 300\sin\theta &= v\sin 67.5°. \end{aligned}$$

Eliminating v between these two equations we get

$$300\cos\theta\,\sin 67.5° - 300\sin\theta\,\cos 67.5° = 100\sin 67.5°,$$

or

$$3\sin(67.5° - \theta) = \sin 67.5°.$$

Therefore, the aircraft should head in direction θ given by

$$\theta = 67.5° - \arcsin\left(\frac{1}{3}\sin 67.5°\right) \approx 49.56°,$$

that is, 49.56° north of east. The groundspeed is now seen to be

$$v = 300\sin\theta / \sin 67.5° \approx 247.15 \text{ km/h}.$$

Thus, the 400 km trip will take about $400/247.15 \approx 1.618$ hours, or about 1 hour and 37 minutes.

Hanging Cables and Chains

When it is suspended from both ends and allowed to hang under gravity, a heavy cable or chain assumes the shape of a **catenary** curve, which is the graph of the hyperbolic cosine function. We will demonstrate this now, using vectors to keep track of the various forces acting on the cable.

Suppose that the cable has line density δ (units of mass per unit length) and hangs as shown in Figure 10.19. Let us choose a coordinate system so that the lowest point L on the cable is at $(0, y_0)$; we will specify the value of y_0 later. If $P = (x, y)$ is another point on the cable, there are three forces acting on the arc LP of the cable between L and P. These are all forces that we can represent using horizontal and vertical components.

(i) The horizontal tension $\mathbf{H} = -H\mathbf{i}$ at L. This is the force that the part of the cable to the left of L exerts on the arc LP at L.

(ii) The tangential tension $\mathbf{T} = T_h\mathbf{i} + T_v\mathbf{j}$. This is the force the part of the cable to the right of P exerts on arc LP at P.

(iii) The weight $\mathbf{W} = -\delta g s\mathbf{j}$ of arc LP, where g is the acceleration of gravity and s is the length of the arc LP.

Since the cable is not moving, these three forces must balance; their vector sum must be zero:

$$\begin{aligned} &\mathbf{T} + \mathbf{H} + \mathbf{W} = \mathbf{0} \\ &(T_h - H)\mathbf{i} + (T_v - \delta g s)\mathbf{j} = \mathbf{0} \end{aligned}$$

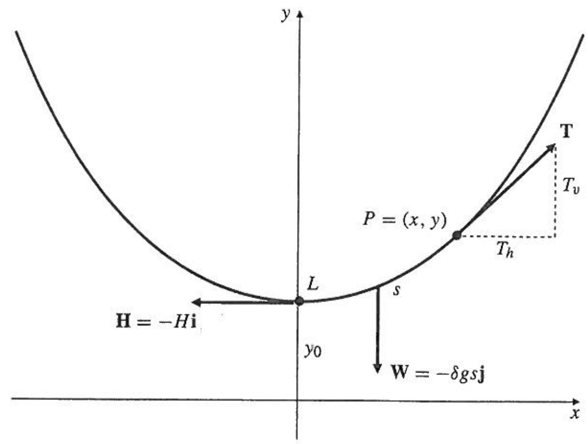

Figure 10.19 A hanging cable and the forces acting on arc LP

Thus $T_h = H$ and $T_v = \delta g s$. Since $\mathbf{T}$ is tangent to the cable at P, the slope of the cable there is

$$\frac{dy}{dx} = \frac{T_v}{T_h} = \frac{\delta g s}{H} = as,$$

where $a = \delta g/H$ is a constant for the given cable. Differentiating with respect to x and using the fact, from our study of arc length, that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

we obtain a second-order differential equation,

$$\frac{d^2y}{dx^2} = a\,\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

to be solved for the equation of the curve along which the hanging cable lies. The appropriate initial conditions are $y = y_0$ and $dy/dx = 0$ at $x = 0$.

Since the differential equation depends on dy/dx rather than y, we substitute $m(x) = dy/dx$ and obtain a first-order equation for m:

$$\frac{dm}{dx} = a\sqrt{1 + m^2}.$$

This equation is separable; we integrate it using the substitution $m = \sinh u$:

$$\int \frac{1}{\sqrt{1+m^2}}\,dm = \int a\,dx$$
$$\int du = \int \frac{\cosh u}{\sqrt{1 + \sinh^2 u}}\,du = ax + C_1$$
$$\sinh^{-1} m = u = ax + C_1$$
$$m = \sinh(ax + C_1).$$

Since $m = dy/dx = 0$ at $x = 0$, we have $0 = \sinh C_1$, so $C_1 = 0$ and

$$\frac{dy}{dx} = m = \sinh(ax).$$

This equation is easily integrated to find y. (Had we used a tangent substitution instead of the hyperbolic sine substitution for m we would have had more trouble here.)

$$y = \frac{1}{a}\cosh(ax) + C_2.$$

If we choose $y_0 = y(0) = 1/a$, then, substituting $x = 0$ we will get $C_2 = 0$. With this choice of y_0, we therefore find that the equation of the curve along which the hanging cable lies is the catenary

$$y = \frac{1}{a}\cosh(ax).$$

Remark If a hanging cable bears loads other than its own weight, it will assume a different shape. For example, a cable supporting a level suspension bridge whose weight per unit length is much greater than that of the cable will assume the shape of a parabola. See Exercise 34 below.

The Dot Product and Projections

There is another operation on vectors in any dimension by which two vectors are combined to produce a number called their *dot product.*

DEFINITION 3

The dot product of two vectors

Given two vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ in $\mathbb{R}^2$, we define their **dot product $\mathbf{u} \bullet \mathbf{v}$** to be the sum of the products of their corresponding components:

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2.$$

The terms **scalar product** and **inner product** are also used in place of dot product. Similarly, for vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in $\mathbb{R}^3$,

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

The dot product has the following algebraic properties, easily checked using the definition above:

$$\begin{aligned}
&\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u} && \text{(commutative law)},\\
&\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w} && \text{(distributive law)},\\
&(t\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (t\mathbf{v}) = t(\mathbf{u} \bullet \mathbf{v}) && \text{(for real } t),\\
&\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2.
\end{aligned}$$

The real significance of the dot product is shown by the following result, which could have been used as the definition of dot product:

THEOREM 1

If θ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ $(0 \le \theta \le \pi)$, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

In particular, $\mathbf{u} \bullet \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. (Of course, the zero vector is perpendicular to every vector.)

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PROOF Refer to Figure 10.20 and apply the Cosine Law to the triangle with the arrows **u**, **v**, and **u** − **v** as sides:

$$\begin{aligned} |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta &= |\mathbf{u}-\mathbf{v}|^2 = (\mathbf{u}-\mathbf{v}) \bullet (\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u} \bullet (\mathbf{u}-\mathbf{v}) - \mathbf{v} \bullet (\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \bullet \mathbf{v} \end{aligned}$$

Hence $|\mathbf{u}||\mathbf{v}|\cos\theta = \mathbf{u} \bullet \mathbf{v}$, as claimed.

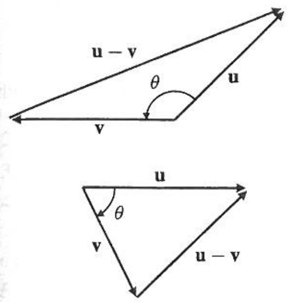

Figure 10.20 Applying the Cosine Law to a triangle reveals the relationship between dot the product and angle between vectors

EXAMPLE 4 Find the angle θ between the vectors $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Solution Solving the formula $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ for θ, we obtain

$$\begin{aligned} \theta = \cos^{-1}\frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} &= \cos^{-1}\left(\frac{(2)(3) + (1)(-2) + (-2)(-1)}{3\sqrt{14}}\right) \\ &= \cos^{-1}\frac{2}{\sqrt{14}} \approx 57.69^\circ. \end{aligned}$$

It is sometimes useful to project one vector along another. We define both scalar and vector projections of **u** in the direction of **v**:

DEFINITION 4

Scalar and vector projections

The **scalar projection** s of any vector **u** in the direction of a nonzero vector **v** is the dot product of **u** with a unit vector in the direction of **v**. Thus, it is the *number*

$$s = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = |\mathbf{u}|\cos\theta,$$

where θ is the angle between **u** and **v**.

The **vector projection**, $\mathbf{u_v}$, of **u** in the direction of **v** (see Figure 10.21) is the scalar multiple of a unit vector $\hat{\mathbf{v}}$ in the direction of **v**, by the scalar projection of **u** in the direction of **v**; that is,

$$\text{vector projection of } \mathbf{u} \text{ along } \mathbf{v} = \mathbf{u_v} = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|}\hat{\mathbf{v}} = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

Note that $|s|$ is the length of the line segment along the line of **v** obtained by dropping perpendiculars to that line from the tail and head of **u**. (See Figure 10.21.) Also, s is negative if $\theta > 90^\circ$.

It is often necessary to express a vector as a sum of two other vectors parallel and perpendicular to a given direction.

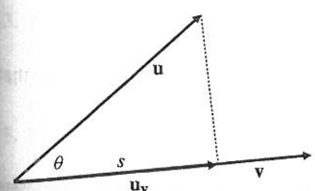

Figure 10.21 The scalar projection s and the vector projection $\mathbf{u_v}$ of vector **u** along vector **v**

EXAMPLE 5 Express the vector $3\mathbf{i} + \mathbf{j}$ as a sum of vectors $\mathbf{u} + \mathbf{v}$, where **u** is parallel to the vector $\mathbf{i} + \mathbf{j}$ and **v** is perpendicular to **u**.

Solution

METHOD I (Using vector projection) Note that **u** must be the vector projection of $3\mathbf{i} + \mathbf{j}$ in the direction of $\mathbf{i} + \mathbf{j}$. Thus,

$$\begin{aligned} \mathbf{u} &= \frac{(3\mathbf{i}+\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}|^2}(\mathbf{i}+\mathbf{j}) = \frac{4}{2}(\mathbf{i}+\mathbf{j}) = 2\mathbf{i}+2\mathbf{j} \\ \mathbf{v} &= 3\mathbf{i}+\mathbf{j}-\mathbf{u} = \mathbf{i}-\mathbf{j}. \end{aligned}$$

METHOD II (From basic principles) Since **u** is parallel to **i**+**j** and **v** is perpendicular to **u**, we have

$$\mathbf{u} = t(\mathbf{i}+\mathbf{j}) \quad \text{and} \quad \mathbf{v} \bullet (\mathbf{i}+\mathbf{j}) = 0,$$

for some scalar t. We want $\mathbf{u}+\mathbf{v} = 3\mathbf{i}+\mathbf{j}$. Take the dot product of this equation with $\mathbf{i}+\mathbf{j}$:

$$\mathbf{u} \bullet (\mathbf{i}+\mathbf{j}) + \mathbf{v} \bullet (\mathbf{i}+\mathbf{j}) = (3\mathbf{i}+\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j})$$
$$t(\mathbf{i}+\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j}) + 0 = 4.$$

Thus $2t = 4$, so $t = 2$. Therefore,

$$\mathbf{u} = 2\mathbf{i}+2\mathbf{j} \quad \text{and} \quad \mathbf{v} = 3\mathbf{i}+\mathbf{j}-\mathbf{u} = \mathbf{i}-\mathbf{j}.$$

Vectors in n-Space

All the above ideas make sense for vectors in spaces of any dimension. Vectors in $\mathbb{R}^n$ can be expressed as linear combinations of the n unit vectors

$\mathbf{e_1}$	from the origin to the point	$(1, 0, 0, \ldots, 0)$
$\mathbf{e_2}$	from the origin to the point	$(0, 1, 0, \ldots, 0)$
	$\vdots$	
$\mathbf{e_n}$	from the origin to the point	$(0, 0, 0, \ldots, 1)$.

These vectors constitute a *standard basis* in $\mathbb{R}^n$. The n-vector $\mathbf{x}$ with components $x_1, x_2, \ldots, x_n$ is expressed in the form

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}.$$

The length of $\mathbf{x}$ is $|\mathbf{x}| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2}$. The angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \cos^{-1} \frac{\mathbf{x} \bullet \mathbf{y}}{|\mathbf{x}||\mathbf{y}|},$$

where

$$\mathbf{x} \bullet \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

We will not make much use of n-vectors for $n > 3$ but you should be aware that everything said up until now for 2-vectors or 3-vectors extends to n-vectors.

EXERCISES 10.2

1. Let $A = (-1, 2)$, $B = (2, 0)$, $C = (1, -3)$, $D = (0, 4)$. Express each of the following vectors as a linear combination of the standard basis vectors **i** and **j** in $\mathbb{R}^2$.

(a) $\overrightarrow{AB}$, (b) $\overrightarrow{BA}$, (c) $\overrightarrow{AC}$, (d) $\overrightarrow{BD}$, (e) $\overrightarrow{DA}$,

(f) $\overrightarrow{AB} - \overrightarrow{BC}$, (g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD}$, and

(h) $\dfrac{\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}}{3}$.

In Exercises 2–3, calculate the following for the given vectors **u** and **v**:

(a) $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}-3\mathbf{v}$,

(b) the lengths $|\mathbf{u}|$ and $|\mathbf{v}|$, lel to $\mathbf{i}+\mathbf{j}$ and $\mathbf{v}$ is perpendicular

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following for the given vectors $\mathbf{u}$

$2\mathbf{u} - 3\mathbf{v}$,

(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

4. Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

7. The diagonals of any parallelogram bisect each other.

8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B? For what values of k is the trip not possible?

12. A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?

13. For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

14. Find the angle between a diagonal of a cube and one of the edges of the cube.

15. Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers.

16. **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

17. Find a unit vector that makes equal angles with the three coordinate axes.

18. Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

19. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20. Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

21. Let $\mathbf{a}$ be a nonzero vector, and let b be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

22. Find two unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

23. Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

24. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

26. Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are arbitrary real numbers.

27. **(The triangle inequality)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

28. (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

29. **(Orthonormal bases)** Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

30. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

31. **(Resolving a vector in perpendicular directions)** If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

32. **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that **u**, **v**, and **r** are position vectors of points U, V, and P, respectively, that **u** is not parallel to **v**, and that P lies in the plane containing the origin, U, and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both **v** and **r** as sums of vectors parallel and perpendicular to **u** as suggested in Exercise 31.

33. Given constants r, s, and t, with $r \neq 0$ and $s \neq 0$, and given a vector **a** satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors **x** and **y**.

Hanging cables

34. **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is $\delta g x$ rather than $\delta g s$, show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

10.3 The Cross Product in 3-Space

There is defined, *in 3-space only*, another kind of product of two vectors called a *cross product* or *vector product*, and denoted $\mathbf{u} \times \mathbf{v}$.

DEFINITION 5

For any vectors **u** and **v** in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying the following three conditions:

(i) $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$,

(ii) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the angle between **u** and **v**, and

(iii) **u**, **v**, and $\mathbf{u} \times \mathbf{v}$ form a right-handed triad.

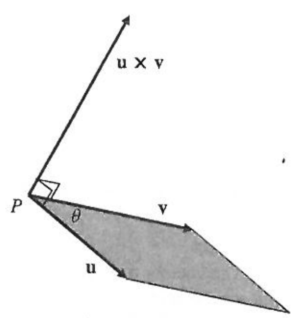

Figure 10.22 $\mathbf{u} \times \mathbf{v}$ is perpendicular to both **u** and **v** and has length equal to the area of the shaded parallelogram

If **u** and **v** are parallel, condition (ii) says that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the zero vector. Otherwise, through any point in $\mathbb{R}^3$ there is a unique straight line that is perpendicular to both **u** and **v**. Condition (i) says that $\mathbf{u} \times \mathbf{v}$ is parallel to this line. Condition (iii) determines which of the two directions along this line is the direction of $\mathbf{u} \times \mathbf{v}$; a right-handed screw advances in the direction of $\mathbf{u} \times \mathbf{v}$ if rotated in the direction from **u** toward **v**. (This is equivalent to saying that the thumb, forefinger, and middle finger of the right hand can be made to point in the directions of **u**, **v**, and $\mathbf{u} \times \mathbf{v}$, respectively.)

If **u** and **v** have their tails at the point P, then $\mathbf{u} \times \mathbf{v}$ is normal (i.e., perpendicular) to the plane through P in which **u** and **v** lie and, by condition (ii), $\mathbf{u} \times \mathbf{v}$ has length equal to the area of the parallelogram spanned by **u** and **v**. (See Figure 10.22.) These properties make the cross product very useful for the description of tangent planes and normal lines to surfaces in $\mathbb{R}^3$.

The definition of cross product given above does not involve any coordinate system and therefore does not directly show the components of the cross product with respect to the standard basis. These components are provided by the following theorem:

THEOREM 2

Components of the cross product

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

PROOF First, we observe that the vector

$$\mathbf{w} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$