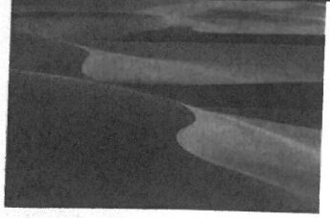
CHAPTER 10



Vectors and Coordinate Geometry in 3-Space

> Lord Ronald said nothing; he flung himself from the room, flung himself upon his horse and rode madly off in all directions. ...
>
> And who is this tall young man who draws nearer to Gertrude with every revolution of the horse? ...
>
> The two were destined to meet. Nearer and nearer they came. And then still nearer. Then for one brief moment they met. As they passed Gertrude raised her head and directed towards the young nobleman two eyes so eye-like in their expression as to be absolutely circular, while Lord Ronald directed towards the occupant of the dogcart a gaze so gaze-like that nothing but a gazelle, or a gas-pipe, could have emulated its intensity.

Stephen Leacock 1869–1944
from *Gertrude the Governess: or, Simple Seventeen*

Introduction

A complete real-variable calculus program involves the study of

(i) real-valued functions of a single real variable,

(ii) vector-valued functions of a single real variable,

(iii) real-valued functions of a real vector variable,

(iv) vector-valued functions of a real vector variable.

Chapters 1–9 are concerned with item (i). The remaining chapters deal with items (ii), (iii), and (iv). Specifically, Chapter 11 deals with vector-valued functions of a single real variable. Chapters 12–14 are concerned with the differentiation and integration of real-valued functions of several real variables, that is, of a real vector variable. Chapters 15 and 16 present aspects of the calculus of functions whose domains and ranges both have dimension greater than one, that is, vector-valued functions of a vector variable. Most of the time we will limit our attention to vector functions with domains and ranges in the plane, or in 3-dimensional space.

In this chapter we will lay the foundation for multivariable and vector calculus by extending the concepts of analytic geometry to three or more dimensions and by introducing vectors as a convenient way of dealing with several variables as a single entity. We also introduce matrices, because these will prove useful for formulating some of the concepts of calculus. This chapter is not intended to be a course in linear algebra. We develop only those aspects that we will use in later chapters and omit most proofs.

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10.1 Analytic Geometry in Three Dimensions

We say that the physical world in which we live is three-dimensional because through any point there can pass three, and no more, straight lines that are **mutually perpendicular**; that is, each of them is perpendicular to the other two. This is equivalent to the fact that we require three numbers to locate a point in space with respect to some reference point (the **origin**). One way to use three numbers to locate a point is by having them represent (signed) distances from the origin, measured in the directions of three mutually perpendicular lines passing through the origin. We call such a set of lines a Cartesian coordinate system, and each of the lines is called a coordinate axis. We usually call these axes the x-axis, the y-axis, and the z-axis, regarding the x- and y-axes as lying in a horizontal plane and the z-axis as vertical. Moreover, the coordinate system should have a **right-handed orientation**. This means that the thumb, forefinger, and middle finger of the right hand can be extended so as to point, respectively, in the directions of the positive x-axis, the positive y-axis, and the positive z-axis. For the more mechanically minded, a right-handed screw will advance in the positive z direction if twisted in the direction of rotation from the positive x-axis toward the positive y-axis. (See Figure 10.1(a).)

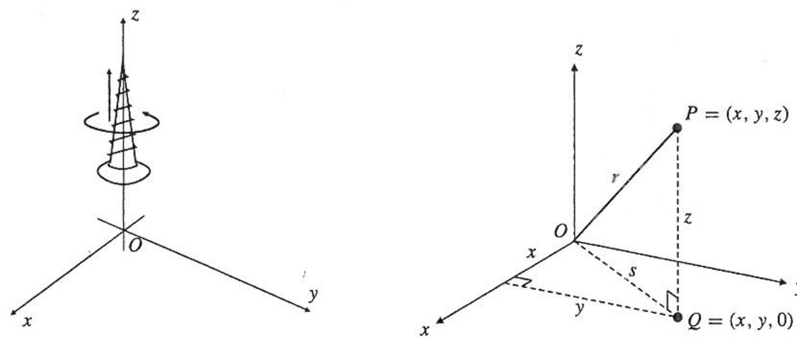

Figure 10.1
(a) The screw moves upward when twisted counterclockwise as seen from above
(b) The three coordinates of a point in 3-space

With respect to such a Cartesian coordinate system, the **coordinates** of a point P in 3-space constitute an ordered triple of real numbers, (x, y, z). The numbers x, y, and z are, respectively, the signed distances of P from the origin, measured in the directions of the x-axis, the y-axis, and the z-axis. (See Figure 10.1(b).)

Let Q be the point with coordinates $(x, y, 0)$. Then Q lies in the xy-plane (the plane containing the x- and y-axes) directly under (or over) P. We say that Q is the vertical projection of P onto the xy-plane. If r is the distance from the origin O to P and s is the distance from O to Q, then, using two right-angled triangles, we have

$$s^2 = x^2 + y^2 \quad \text{and} \quad r^2 = s^2 + z^2 = x^2 + y^2 + z^2.$$

Thus, the distance from P to the origin is given by

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Similarly, the distance r between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ (see Figure 10.2) is

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

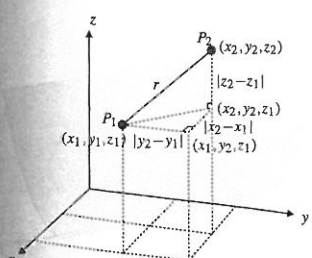

Figure 10.2 Distance between points

EXAMPLE 1 Show that the triangle with vertices $A = (1, -1, 2)$, $B = (3, 3, 8)$, and $C = (2, 0, 1)$ has a right angle.

Solution We calculate the lengths of the three sides of the triangle:

$$a = |BC| = \sqrt{(2-3)^2 + (0-3)^2 + (1-8)^2} = \sqrt{59}$$
$$b = |AC| = \sqrt{(2-1)^2 + (0+1)^2 + (1-2)^2} = \sqrt{3}$$
$$c = |AB| = \sqrt{(3-1)^2 + (3+1)^2 + (8-2)^2} = \sqrt{56}$$

By the cosine law, $a^2 = b^2 + c^2 - 2bc\cos A$. In this case $a^2 = 59 = 3 + 56 = b^2 + c^2$, so that $2bc\cos A$ must be 0. Therefore, $\cos A = 0$ and $A = 90°$.

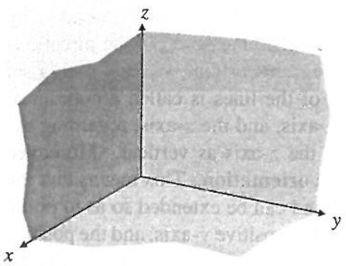

Figure 10.3 The first octant

Just as the x- and y-axes divide the xy-plane into four quadrants, so also the three **coordinate planes** in 3-space (the xy-plane, the xz-plane, and the yz-plane) divide 3-space into eight **octants**. We call the octant in which $x \geq 0$, $y \geq 0$, and $z \geq 0$ the **first octant**. When drawing graphs in 3-space it is sometimes easier to draw only the part lying in the first octant (Figure 10.3).

An equation or inequality involving the three variables x, y, and z defines a subset of points in 3-space whose coordinates satisfy the equation or inequality. A single equation usually represents a surface (a two-dimensional object) in 3-space.

EXAMPLE 2 **(Some equations and the surfaces they represent)**

(a) The equation $z = 0$ represents all points with coordinates $(x, y, 0)$, that is, the xy-plane. The equation $z = -2$ represents all points with coordinates $(x, y, -2)$, that is, the horizontal plane passing through the point $(0, 0, -2)$ on the z-axis.

(b) The equation $x = y$ represents all points with coordinates (x, x, z). This is a vertical plane containing the straight line with equation $x = y$ in the xy-plane. The plane also contains the z-axis. (See Figure 10.4.)

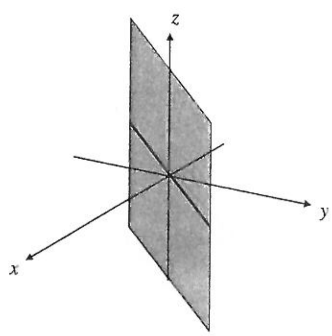

Figure 10.4 Equation $x = y$ defines a vertical plane

(c) The equation $x + y + z = 1$ represents all points the sum of whose coordinates is 1. This set is a plane that passes through the three points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. These points are not collinear (they do not lie on a straight line), so there is only one plane passing through all three. (See Figure 10.5.) The equation $x + y + z = 0$ represents a plane parallel to the one with equation $x + y + z = 1$ but passing through the origin.

(d) The equation $x^2 + y^2 = 4$ represents all points on the vertical circular cylinder containing the circle with equation $x^2 + y^2 = 4$ in the xy-plane. This cylinder has radius 2 and axis along the z-axis. (See Figure 10.6.)

(e) The equation $z = x^2$ represents all points with coordinates (x, y, x^2). This surface is a parabolic cylinder tangent to the xy-plane along the y-axis. (See Figure 10.7.)

(f) The equation $x^2 + y^2 + z^2 = 25$ represents all points (x, y, z) at distance 5 from the origin. This set of points is a *sphere* of radius 5 centred at the origin.

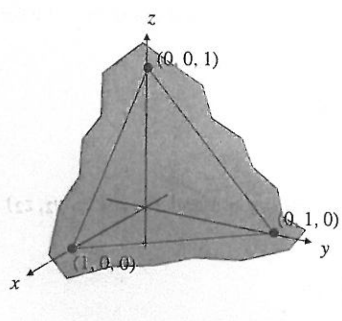

Figure 10.5 The plane with equation $x + y + z = 1$

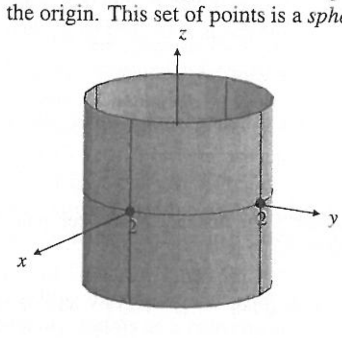

Figure 10.6 The circular cylinder with equation $x^2 + y^2 = 4$

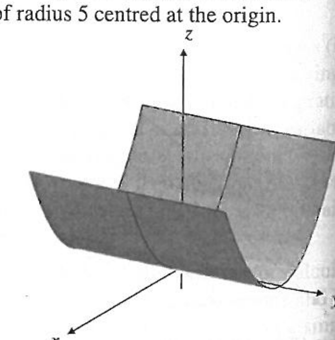

Figure 10.7 The parabolic cylinder with equation $z = x^2$

of the triangle:

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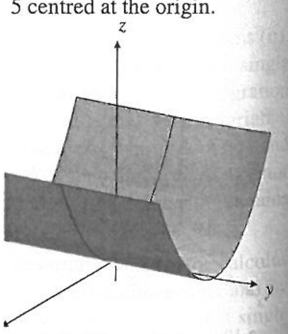


10.7 The parabolic cylinder
quation $z = x^2$

Observe that equations in x, y, and z need not involve each variable explicitly. When one of the variables is missing from the equation, the equation represents a surface *parallel to* the axis of the missing variable. Such a surface may be a plane or a cylinder. For example, if z is absent from the equation, the equation represents in 3-space a vertical (i.e., parallel to the z-axis) surface containing the curve with the same equation in the xy-plane.

Occasionally, a single equation may not represent a two-dimensional object (a surface). It can represent a one-dimensional object (a line or curve), a zero-dimensional object (one or more points), or even nothing at all.

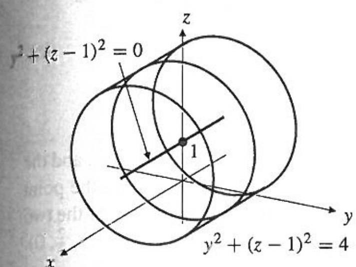


Figure 10.8 The cylinder $y^2 + (z-1)^2 = 4$ and its axial line $y^2 + (z-1)^2 = 0$

EXAMPLE 3 Identify the graphs of: (a) $y^2 + (z-1)^2 = 4$, (b) $y^2 + (z-1)^2 = 0$, (c) $x^2 + y^2 + z^2 = 0$, and (d) $x^2 + y^2 + z^2 = -1$.

Solution

(a) Since x is absent, the equation $y^2 + (z-1)^2 = 4$ represents an object parallel to the x-axis. In the yz-plane the equation represents a circle of radius 2 centred at $(y, z) = (0, 1)$. In 3-space it represents a horizontal circular cylinder, parallel to the x-axis, with axis one unit above the x-axis. (See Figure 10.8.)

(b) Since squares cannot be negative, the equation $y^2 + (z-1)^2 = 0$ implies that $y = 0$ and $z = 1$, so it represents points $(x, 0, 1)$. All these points lie on the line parallel to the x-axis and one unit above it. (See Figure 10.8.)

(c) As in part (b), $x^2 + y^2 + z^2 = 0$ implies that $x = 0$, $y = 0$, and $z = 0$. The equation represents only one point, the origin.

(d) The equation $x^2 + y^2 + z^2 = -1$ is not satisfied by any real numbers x, y, and z, so it represents no points at all.

A single inequality in x, y, and z typically represents points lying on one side of the surface represented by the corresponding equation (together with points on the surface if the inequality is not strict).

EXAMPLE 4 (a) The inequality $z > 0$ represents all points above the xy-plane.

(b) The inequality $x^2 + y^2 \geq 4$ says that the square of the distance from (x, y, z) to the nearest point $(0, 0, z)$ on the z-axis is at least 4. This inequality represents all points lying on or outside the cylinder of Example 2(d).

(c) The inequality $x^2 + y^2 + z^2 \leq 25$ says that the square of the distance from (x, y, z) to the origin is no greater than 25. It represents the solid ball of radius 5 centred at the origin, which consists of all points lying inside or on the sphere of Example 2(f).

Two equations in x, y, and z normally represent a one-dimensional object, the line or curve along which the two surfaces represented by the two equations intersect. Any point whose coordinates satisfy both equations must lie on both the surfaces, so must lie on their intersection.

EXAMPLE 5 What sets of points in 3-space are represented by the following pairs of equations?

$$\text{(a)} \quad \begin{cases} x + y + z = 1 \\ y - 2x = 0 \end{cases} \qquad \text{(b)} \quad \begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y = 1 \end{cases}$$

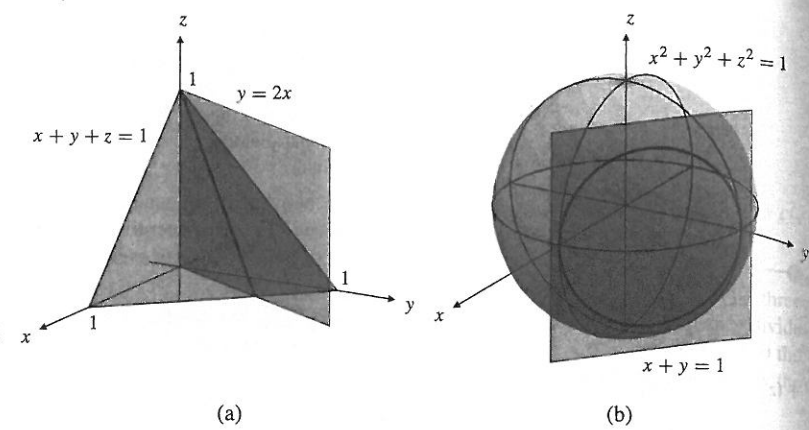


Figure 10.9

(a) The two planes intersect in a straight line

(b) The plane intersects the sphere in a circle

Solution

(a) The equation $x + y + z = 1$ represents the oblique plane of Example 2(c), and the equation $y - 2x = 0$ represents a vertical plane through the origin and the point $(1, 2, 0)$. Together these two equations represent the line of intersection of the two planes. This line passes through, for example, the points $(0, 0, 1)$ and $(\frac{1}{3}, \frac{2}{3}, 0)$. (See Figure 10.9(a).)

(b) The equation $x^2 + y^2 + z^2 = 1$ represents a sphere of radius 1 with centre at the origin, and $x + y = 1$ represents a vertical plane through the points $(1, 0, 0)$ and $(0, 1, 0)$. The two surfaces intersect in a circle, as shown in Figure 10.9(b). The line from $(1, 0, 0)$ to $(0, 1, 0)$ is a diameter of the circle, so the centre of the circle is $(\frac{1}{2}, \frac{1}{2}, 0)$, and its radius is $\sqrt{2}/2$.

In Sections 10.4 and 10.5 we will see many more examples of geometric objects in 3-space represented by simple equations.

Euclidean *n*-Space

Mathematicians and users of mathematics frequently need to consider ***n*-dimensional space**, where n is greater than 3 and may even be infinite. It is difficult to visualize a space of dimension 4 or higher geometrically. The secret to dealing with these spaces is to regard the points in n-space as *being* ordered n-tuples of real numbers; that is, $(x_1, x_2, \ldots, x_n)$ is a point in n-space instead of just being the coordinates of such a point. We stop thinking of points as existing in physical space and start thinking of them as algebraic objects. We usually denote n-space by the symbol $\mathbb{R}^n$ to show that its points are n-tuples of *real* numbers. Thus $\mathbb{R}^2$ and $\mathbb{R}^3$ denote the plane and 3-space, respectively. Note that in passing from $\mathbb{R}^3$ to $\mathbb{R}^n$ we have altered the notation a bit: in $\mathbb{R}^3$ we called the coordinates x, y, and z, while in $\mathbb{R}^n$ we called them $x_1, x_2, \ldots$ and x_n so as not to run out of letters. We could, of course, talk about coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$ and (x_1, x_2) in the plane $\mathbb{R}^2$, but (x, y, z) and (x, y) are traditionally used there.

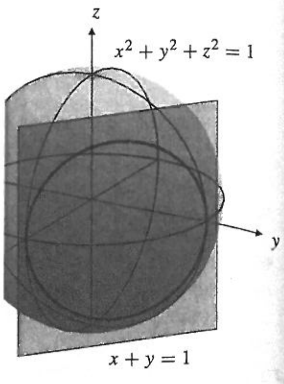
Although we think of points in $\mathbb{R}^n$ as n-tuples rather than geometric objects, we do not want to lose all sight of the underlying geometry. By analogy with the two- and three-dimensional cases, we still consider the quantity

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

as representing the *distance* between the points with coordinates $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. Also, we call the $(n - 1)$-dimensional set of points in $\mathbb{R}^n$ that satisfy the equation $x_n = 0$ a **hyperplane**, by analogy with the plane $z = 0$ in $\mathbb{R}^3$.

Describing Sets in the Plane, 3-Space, and *n*-Space

We conclude this section by collecting some definitions of terms used to describe sets



(b)

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n-Space

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of points in $\mathbb{R}^n$ for $n \geq 2$. These terms belong to the branch of mathematics called **topology**, and they generalize the notions of open and closed intervals and endpoints used to describe sets on the real line $\mathbb{R}$. We state the definitions for $\mathbb{R}^n$, but we are most interested in the cases where $n = 2$ or $n = 3$.

A **neighbourhood** of a point P in $\mathbb{R}^n$ is a set of the form

$$B_r(P) = \{Q \in \mathbb{R}^n : \text{distance from } Q \text{ to } P < r\}$$

for some $r > 0$.

For $n = 1$, if $p \in \mathbb{R}$, then $B_r(p)$ is the **open interval** $(p - r, p + r)$ centred at p.

For $n = 2$, $B_r(P)$ is the **open disk** of radius r centred at point P.

For $n = 3$, $B_r(P)$ is the **open ball** of radius r centred at point P.

A set S is **open** in $\mathbb{R}^n$ if every point of S has a neighbourhood contained in S. Every neighbourhood is itself an open set. Other examples of open sets in $\mathbb{R}^2$ include the sets of points (x, y) such that $x > 0$, or such that $y > x^2$, or even such that $y \neq x^2$. Typically, sets defined by strict inequalities (using $>$ and $<$) are open. Examples in $\mathbb{R}^3$ include the sets of points (x, y, z) satisfying $x + y + z > 2$, or $1 < x < 3$.

The whole space $\mathbb{R}^n$ is an open set in itself. For technical reasons, the empty set (containing no points) is also considered to be open. (No point in the empty set fails to have a neighbourhood contained in the empty set.)

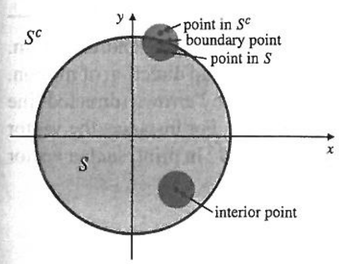

Figure 10.10 The closed disk S consisting of points $(x, y) \in \mathbb{R}^2$ that satisfy $x^2 + y^2 \leq 1$. Note the shaded neighbourhoods of the boundary point and the interior point.
bdry(S) is the circle $x^2 + y^2 = 1$
int(S) is the open disk $x^2 + y^2 < 1$
ext(S) is the open set $x^2 + y^2 > 1$

The **complement**, S^c, of a set S in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ that do not belong to S. For example, the complement of the set of points (x, y) in $\mathbb{R}^2$ such that $x > 0$ is the set of points for which $x \leq 0$. A set is said to be **closed** if its complement is open. Typically, sets defined by nonstrict inequalities (using $\geq$ and $\leq$) are closed. Closed intervals are closed sets in $\mathbb{R}$. Since the whole space and the empty set are both open in $\mathbb{R}^n$ and are complements of each other, they are also both closed. They are the only sets that are both open and closed.

A point P is called a **boundary point** of a set S if every neighbourhood of P contains both points in S and points in S^c. The **boundary**, bdry(S), of a set S is the set of all boundary points of S. For example, the boundary of the closed disk $x^2 + y^2 \leq 1$ in $\mathbb{R}^2$ is the circle $x^2 + y^2 = 1$. A closed set contains all its boundary points. An open set contains none of its boundary points.

A point P is an **interior point** of a set S if it belongs to S but not to the boundary of S. P is an **exterior point** of S if it belongs to the complement of S but not to the boundary of S. The **interior**, int(S), and **exterior**, ext(S), of S consist of all the interior points and exterior points of S, respectively. Both int(S) and ext(S) are open sets. If S is open, then int(S) $= S$. If S is closed, then ext(S) $= S^c$. See Figure 10.10.

EXERCISES 10.1

Find the distance between the pairs of points in Exercises 1–4.

1. (0, 0, 0) and (2, −1, −2)

2. (−1, −1, −1) and (1, 1, 1)

3. (1, 1, 0) and (0, 2, −2)

4. (3, 8, −1) and (−2, 3, −6)

5. What is the shortest distance from the point (x, y, z) to (a) the xy-plane? (b) the x-axis?

6. Show that the triangle with vertices (1, 2, 3), (4, 0, 5), and (3, 6, 4) has a right angle.

7. Find the angle A in the triangle with vertices $A = (2, -1, -1)$, $B = (0, 1, -2)$, and $C = (1, -3, 1)$.

8. Show that the triangle with vertices (1, 2, 3), (1, 3, 4), and (0, 3, 3) is equilateral.

9. Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1), and (0, 1, 1).

10. What is the distance from the origin to the point $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$?

11. What is the distance from the point $(1, 1, \ldots, 1)$ in n-space to the closest point on the x_1-axis?

In Exercises 12–23, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

12. $z = 2$

13. $y \geq -1$

14. $z = x$

15. $x + y = 1$

16. $x^2 + y^2 + z^2 = 4$

17. $(x-1)^2+(y+2)^2+(z-3)^2=4$

18. $x^2+y^2+z^2=2z$ **19.** $y^2+z^2 \le 4$

20. $x^2+z^2=4$ **21.** $z=y^2$

22. $z \ge \sqrt{x^2+y^2}$ **23.** $x+2y+3z=6$

In Exercises 24–32, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

24. $\begin{cases} x=1 \\ y=2 \end{cases}$ **25.** $\begin{cases} x=1 \\ y=z \end{cases}$

26. $\begin{cases} x^2+y^2+z^2=4 \\ z=1 \end{cases}$ **27.** $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2+z^2=4x \end{cases}$

28. $\begin{cases} x^2+y^2+z^2=4 \\ x^2+z^2=1 \end{cases}$ **29.** $\begin{cases} x^2+y^2=1 \\ z=x \end{cases}$

30. $\begin{cases} y \ge x \\ z \le y \end{cases}$ **31.** $\begin{cases} x^2+y^2 \le 1 \\ z \ge y \end{cases}$

32. $\begin{cases} x^2+y^2+z^2 \le 1 \\ \sqrt{x^2+y^2} \le z \end{cases}$

In Exercises 33–36, specify the boundary and the interior of the plane sets S whose points (x, y) satisfy the given conditions. Is S open, closed, or neither?

33. $0 < x^2+y^2 < 1$ **34.** $x \ge 0, \quad y < 0$

35. $x+y=1$ **36.** $|x|+|y| \le 1$

In Exercises 37–40, specify the boundary and the interior of the sets S in 3-space whose points (x, y, z) satisfy the given conditions. Is S open, closed, or neither?

37. $1 \le x^2+y^2+z^2 \le 4$ **38.** $x \ge 0, \quad y > 1, \quad z < 2$

39. $(x-z)^2+(y-z)^2=0$ **40.** $x^2+y^2<1, \; y+z>2$

10.2 Vectors

A **vector** is a quantity that involves both **magnitude** (size or length) and **direction**. For instance, the *velocity* of a moving object involves its speed and direction of motion, so is a vector. Such quantities are represented geometrically by arrows (directed line segments) and are often actually identified with these arrows. For instance, the vector $\overrightarrow{AB}$ is an arrow with tail at the point A and head at the point B. In print, such a vector is usually denoted by a single letter in boldface type,

$$\mathbf{v} = \overrightarrow{AB}.$$

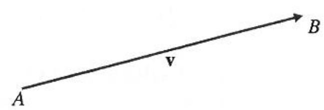

Figure 10.11 The vector $\mathbf{v} = \overrightarrow{AB}$

(See Figure 10.11.) In handwriting, an arrow over a letter ($\vec{v} = \overrightarrow{AB}$) can be used to denote a vector. The *magnitude* of the vector $\mathbf{v}$ is the length of the arrow and is denoted $|\mathbf{v}|$ or $|\overrightarrow{AB}|$.

While vectors have magnitude and direction, they do not generally have *position*; that is, they are not regarded as being in a particular place. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are considered *equal* if they have *the same length and the same direction*, even if their representative arrows do not coincide. The arrows must be parallel, have the same length, and point in the same direction. In Figure 10.12, for example, if $ABYX$ is a parallelogram, then $\overrightarrow{AB} = \overrightarrow{XY}$.

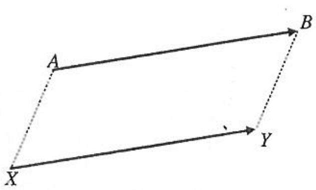

Figure 10.12 $\overrightarrow{AB} = \overrightarrow{XY}$

For the moment, we consider plane vectors, that is, vectors whose representative arrows lie in a plane. If we introduce a Cartesian coordinate system into the plane, we can talk about the x and y components of any vector. If $A=(a,b)$ and $P=(p,q)$, as shown in Figure 10.13, then the x and y components of $\overrightarrow{AP}$ are, respectively, $p-a$ and $q-b$. Note that if O is the origin and X is the point $(p-a, q-b)$, then

$$|\overrightarrow{AP}| = \sqrt{(p-a)^2+(q-b)^2} = |\overrightarrow{OX}|$$

$$\text{slope of } \overrightarrow{AP} = \frac{q-b}{p-a} = \text{slope of } \overrightarrow{OX}.$$

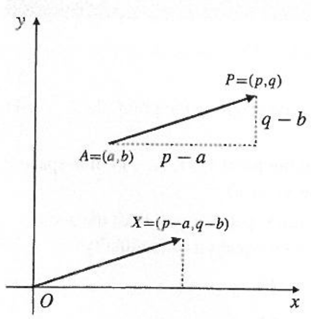

Figure 10.13 Components of a vector

Hence $\overrightarrow{AP} = \overrightarrow{OX}$. In general, two vectors are equal if and only if they have the same x components and y components.

There are two important algebraic operations defined for vectors: addition and scalar multiplication.