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FRONT PROPAGATION IN A NOISY, NONSMOOTH, EXCITABLE MEDIUM

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Abstract. We consider the impact of noise on the stability and propagation of fronts in an excitable media with a piece-wise smooth, discontinuous ignition process. In a neighborhood of the ignition threshold the system interacts strongly with noise, the front can loose monotonicity, resulting in multiple crossings of the ignition threshold. We adapt the renormalization group methods developed for coherent structure interaction, a key step being to determine pairs of function spaces for which the the ignition function is Frechet differentiable, but for which the associated semi-group, $S(t)$, is integrable at $t = 0$. We parameterize a neighborhood of the front solution through a dynamic front position and a co-dimension one remainder. The front evolution and the asymptotic decay of the remainder are on the same time scale, the RG approach shows that the remainder becomes asymptotically small, in terms of the noise strength and regularity, and the front propagation is driven by a competition between the ignition process and the noise.

1. **Introduction.** Non-smooth excitable systems arise as models of combustion [7], nerve impulses [21],[8], elastic displacements during phase transitions [2], and proton conduction in weakly hydrated polymer electrolyte membranes (PEMs), [9]. These systems often support front solutions which correspond to moving phase transitions. Typically the non-smoothness arises as an approximation to a system which is so sensitive to changes in state at critical values of internal variables, for example changes in temperature near the flash point of a mixture, that its response may be considered discontinuous. We consider a singularly perturbed, piece-wise smooth system with a discontinuity at an ignition threshold for which the noiseless system supports a unique, stable front solution. At spatial points where the front crosses criticality the system will be particularly sensitive to noise, leading the solution to lose monotonicity, and engendering multiple crossings of the ignition threshold. We identify regimes in which the front preserves its stability, showing that noise smears the front location and modifies the front propagation.

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Front propagation in nonsmooth, noisy systems has received considerable attention. The Fisher-Kolmogorov-Petrowskii-Piscounov equation with cut-off was introduced in [5] to model an N-particle system in which the reaction term is set to zero for concentrations below a small threshold. The impact of noise on front velocity and stability has been studied in [13] and [6] in the limit of vanishing threshold size. In [20] front propagation was studied in excitable, singularly perturbed reaction-diffusion systems subject to stochastic noise, showing that in a particular scaling the noise can lead to self-induced stochastic resonance. However this study restricted the noise to a neighborhood of the origin, avoiding the complication of the noise interacting with the front's stability and its propagation. The present work addresses precisely this point, adding a noise term which lies in $L^\infty(\mathbb{R}_+, H^{-\rho})$ for $\rho \in (0, \frac{3}{2})$, to a reaction-diffusion system, (1), with a discontinuous reaction term and a front solution whose width scales like $O(\sqrt{\epsilon})$. The main result, Theorem 1, shows that the front retains its stability, but the ignition point smears into an "ignition set" over which the front can have multiple crossings of the ignition threshold. Moreover in the scalings for which we show the front retains its stability, the ignition set is thinner than the front and the impact of the noise on the front velocity is a correction to the noiseless front velocity.

In section 2 we present the model equation, derive the asymptotic form for a quasi-traveling front solution of the noiseless equation. In section 3 we present the rescaled system, present the main result, Theorem 1, show that in appropriate function spaces the noise-less system is Frechet differentiable about the front solution, and determine the spectrum of the linearized operator. In section 4 we show the linear operator generates an analytic semigroup with an integrable singularity at $t = 0$, subject to additional constraints on the function spaces. Finally in section 5, we prove Theorem 1, adapting the renormalization group (RG) approach developed in [14] for interactions of coherent structures, and extended to a variety of singularly perturbed settings in [1, 22, 23], to prove the stability of the front solution in the noisy environment, and derive an asymptotic expression for the front evolution. The RG approach has the advantage over an invariant manifold approach, [18, 19, 24, 3] in that an exact traveling wave form is not required, rather an ansatz which, in some asymptotic sense, satisfies the traveling wave equation for the underlying system.

2. **Model description and construction of traveling ansatz.**

Recent experimental work, [10], has shown that autohumidified PEM fuel cells fed from dry inlet gases can exhibit hysteresis, slow transients, and long period relaxation oscillations. Subsequently a model was proposed, [9], accounting for the impact of membrane water content on protonic conductivity which suggests that the fuel cell system possesses two stable states, an ignited state in which the membrane has sufficient water to sustain the electrochemical reaction, and an extinguished state characterized by high membrane protonic resistance and low water content. The slow transients and hysteresis could then be explained by traveling "ignition" waves which move laterally along the membrane. In this context, we consider a thin 2D membrane exposed on one side to air and the other side to gaseous fuel, both at prescribed humidity, which varies with position along the length of the membrane. The relative humidity of the air produces a local reference water content for the membrane, which is modified by the local production of water by the electrochemical reaction, which occurs only in the regions of the membrane which are already sufficiently humidified. Averaging the water content in the through-plane direction of the membrane, we consider a simplified model for the diffusive transport of water content, $U(y,t)$ within the plane of the membrane

$$U_t = F(U) + \epsilon^p \eta(y,t), \tag{1}$$

where

$$F(U) = \epsilon U_{yy} - (U - g(y)) + \mu\sigma(U). \tag{2}$$

Here y is scaled length lateral to the membrane and t is scaled time. The nonlinear ignition functional σ models the conductivity of membrane and possesses a jump discontinuity at the ignition threshold, σ_c,

$$\sigma(s) = \begin{cases} 0 & s < \sigma_c, \\ s & s > \sigma_c. \end{cases} \tag{3}$$

As the membrane absorbs water it develops a network of pores, at a critical concentration of water the pores percolate and the membrane becomes protonically conductive, permitting the water producing reaction to "ignite." The parameter $\mu \in (0,1)$ governs the strength of the water production from the electrochemical reaction. The membrane exchanges water with the gas channels which run laterally on either side of the membrane. The local relative humidity of the gas channels determines an equilibrium water content within the membrane, $g(y) \in \mathcal{C}^2(\mathbb{R})$, which we take to be an ϵ independent, monotonically increasing function of y, approaching the limits $g(y) \to g_\pm$ as $y \to \pm\infty$. Condensation of liquid droplets within the channels produce local elevations of membrane water content. We model this through the noise term, $\eta \in L^\infty(\mathbf{R}^+, H^{-\rho})$, where $\rho \in \left(0, \frac{3}{2}\right)$. Within the context of the application η is viewed as a series of space and time uncorrelated delta functions, although the results obtained here apply to the more general choices of η. The lateral diffusivity coefficient $\epsilon \ll 1$, and is balanced against the strength of the noise, which is scaled as ϵ^p.

The noiseless problem, (1) with $\eta = 0$, possesses a family of traveling fronts, parameterized by their front position y_0, whose construction is complicated by the variation of the equilibrium function, g, with lateral distance y. However the RG approach and the rigorous analysis which follows does not require an exact traveling wave solution, rather we construct a leading order asymptotic form for the traveling wave, which serves as a pre-conditioner for the development of the coordinate systems used in the rigorous RG analysis.

Lemma 1. *For each front position $y_0 \in \mathcal{K} = (\kappa_l, \kappa_r)$, defined in (6), there exists a unique velocity $v = v(y_0)$, varying smoothly in y_0, such that the composite traveling wave ansatz $\phi(y; y_0)$ given in (7), is $\mathcal{C}^1(\mathbb{R})$ and asymptotically satisfies the deterministic problem, (1) with $\eta = 0$, with residual*

$$F(\phi) = -v\sqrt{\epsilon}\frac{\partial \phi_i}{\partial y} + \epsilon\left[\frac{\partial^2 \phi_o}{\partial y^2}\right], \tag{4}$$

where $\left[\frac{\partial^2\phi_o}{\partial y^2}\right]$ denotes the piecewise second derivative of the outer solution (5). Moreover the set $\mathcal{K}$ is forward invariant under the flow $y_0' = v(y_0)$ since if either $\kappa_r < \infty$ or $\kappa_l > -\infty$ then $v(\kappa_r) < 0$ and $v(\kappa_l) > 0$.

Proof: The outer problem, (1) with $\epsilon = 0$, possesses a one parameter family of steady front solutions

$$\phi_o = \begin{cases} g(y) & y < y_0, \\ \dfrac{g(y)}{1-\mu} & y > y_0, \end{cases} \tag{5}$$

as long as $y_0 \in \mathcal{K}$ given by

$$\mathcal{K} = (\kappa_l, \kappa_r) \equiv \left\{ y_0 \in \mathbb{R} \middle| g(y_0) < \sigma_c < \frac{g(y_0)}{1-\mu} \right\}. \tag{6}$$

We connect these to an inner solution, ϕ_i, in the stretched variable $\xi = \frac{y-y_0}{\sqrt{\epsilon}}$. Looking for a quasi-steady travelling wave solution $U(\xi, t) = \phi_i(\xi - vt; y_0)$ of (1), ϕ_i satisfies

$$\phi_i'' + v\phi_i' - (\phi_i - g(y_0)) + \mu\sigma(\phi_i) = 0,$$

with the boundary conditions $\phi_i \to g(y_0)$ as $\xi \to -\infty$ and $\phi_i \to \frac{g(y_0)}{1-\mu}$ as $\xi \to \infty$. The inner equation can be written as a piece-wise linear second order system with two saddle fixed points $(g(y_0), 0)$ and $(g(y_0)/(1-\mu), 0)$. The inner solution is obtained by connecting the unstable manifold of the left saddle point with the stable manifold of the right one; the composite solution is obtained by patching the inner and outer solution together, yielding

$$\phi = \begin{cases} g(y) + (\sigma_c - g(y_0))e^{\lambda_1^+(y-y_0)/\sqrt{\epsilon}} & y < y_0, \\ \frac{g(y)}{1-\mu} + (\sigma_c - \frac{g(y_0)}{1-\mu})e^{\lambda_2^-(y-y_0)/\sqrt{\epsilon}} & y > y_0, \end{cases} \tag{7}$$

where $\lambda_1^+(v) = \frac{1}{2}\left(-v + \sqrt{v^2+4}\right)$ and $\lambda_2^-(v) = -\frac{1}{2}\left(v + \sqrt{v^2 + 4(1-\mu)}\right)$ are the v dependent unstable and stable eigenvalues of the left and right saddle points, respectively. We render $\phi \in \mathcal{C}^1$ by requiring the continuity of ϕ' at $y = y_0$. The jump in the derivative is given by,

$$J(v) = \left(\left(\sigma_c - \frac{g(y_0)}{1-\mu}\right) \frac{\lambda_2^-(v)}{\sqrt{\epsilon}} + \frac{g'(y_0)}{1-\mu} \right) - \left((\sigma_c - g(y_0)) \frac{\lambda_1^+(v)}{\sqrt{\epsilon}} + g'(y_0) \right),$$

and the condition $J(v) = 0$ determines $v = v(y_0)$ uniquely since J is strictly increasing in v, and satisfies $vJ(v) > 0$ for v sufficiently large. The form of the residual, (4) follows from the construction of the inner and outer solutions, and the piecewise linearity of equation (1). To see that $\mathcal{K}$ is forward invariant under the flow, note that if κ_r is finite, then $g(\kappa_r) = \sigma_c$ and thus $J(v) = 0$ implies $\lambda_2^-(v) = \mathcal{O}(\sqrt{\epsilon})$ and hence $v < 0$. A similar arguement holds if $\kappa_l > -\infty$. □

3. The rescaled problem and its linearization.

The front ϕ changes its form as it evolves. In the RG approach we freeze the coordinate system, determined by the front shape and speed, so as to work with a constant coefficient linear problem, and then update the frozen coordinates when the secularity, as measured by the difference between the frozen and the evolving fronts, grows too large. For a particular frozen front location, $\overline{y}_0$, at time $t = t_0$, we shift to a co-moving frame with fixed velocity

$$z = \frac{y - \overline{y}_0 - \sqrt{\epsilon}v(\overline{y}_0)(t-t_0)}{\sqrt{\epsilon}}, \tag{8}$$

and the time dependent front location in the convective frame is given by,

$$z_0 = \frac{y_0 - \overline{y}_0 - \sqrt{\epsilon}v(\overline{y}_0)(t-t_0)}{\sqrt{\epsilon}}, \tag{9}$$

We denote $\bar{z}_0 \equiv 0$, $\bar{v} \equiv v(\bar{z}_0)$ and $\bar{\phi} \equiv \phi(z; \bar{z}_0)$ while $'$ denotes derivative w.r.t. z. Under the transformation (8), the equation (1) takes the form

$$U_t = F(U) = U_{zz} + vU_z - (U - g(z)) + \mu\sigma(U) + \epsilon^{p_s}\eta_s(z,t), \tag{10}$$

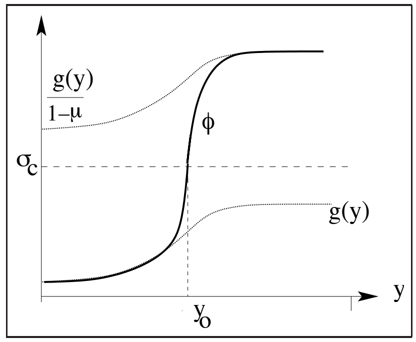

FIGURE 1. The composite solution ϕ.

where $\eta_s(z,t)$ is the rescaled noise. The rescaling of the functions in $H^{-\rho}$ increases their amplitude. Indeed the Fourier transform of functions $\eta \in H^{-\rho}$ has, generically, the asymptotic form $\hat{\eta}(k) \approx c|k|^{\rho-\frac{1}{2}}$ for $|k| \gg 1$, and hence

$$\hat{\eta}_s(k) = \frac{1}{\sqrt{\epsilon}}\hat{\eta}\left(\frac{k}{\sqrt{\epsilon}}\right) \approx \epsilon^{-(\frac{\rho}{2}+\frac{1}{4})}|k|^{\rho-\frac{1}{2}},$$

for $|k| \gg 1$. We absorb this rescaling into the ϵ coefficient, defining $p_s = p - \frac{\rho}{2} - \frac{1}{4}$, so that $\|\eta_s(t)\|_{H_s^{-\rho}} \leq 1$.

Our main result, stated below, shows that the solutions of (1) which originate from a neighborhood of the front ansatz can be decomposed as a front and a remainder which remains small in the scaled H^γ norm,

$$\|W\|_{H_s^\gamma} = \left\| W\left(\frac{\cdot}{\sqrt{\epsilon}}\right)\right\|_{H^\gamma},$$

which, for $\gamma > \frac{1}{2}$, controls the L^∞ norm, uniformly in $\epsilon > 0$. For emphasis we also introduce the inner product in the fast variables

$$(f,g)_{L_s^2} = \int_{\mathbb{R}} f(z)g(z)dz,$$

where z is given by (8).

Theorem 1. *Fix $\rho \in \left(0, \frac{3}{2}\right)$, $p > \frac{\rho}{2} + \frac{1}{4}$ and let $\gamma \in (\frac{1}{2}, 1)$ satisfy $\gamma < 2 - \rho$ and $\gamma < \frac{1}{p-\frac{\rho}{2}-\frac{1}{4}} - \frac{3}{2}$ and let $\alpha_0 \in \left(0, \frac{\gamma-1/2}{\gamma+1/2}\right)$ be arbitrary. Then there exists $\nu > 0$ and $M > 1$ such that for all $\epsilon \in (0, \epsilon_0)$, with $\epsilon_0 = M^{-\frac{1}{\alpha_0(p-\rho/2-1/4)}}$, and for all initial data of the form*

$$U_0(y,t) = \phi(y; y_*) + W_0(y,t),$$

where ϕ is as constructed in Lemma 1, $y_* \in \mathcal{K}$ is given by (6), and

$$\|W_0\|_{H_s^\gamma} \leq M\epsilon^{\alpha_0(p-\rho/2-1/4)(\gamma+1/2)},$$

the solution of the governing equation (1) can be decomposed as

$$U(y,t) = \phi(y; y_0(t)) + W(y,t),$$

for all $t > 0$ while the remainder W satisfies

$$\|W\|_{H_s^\gamma} \leq M\left(e^{-\nu(t-t_0)}\|W_0\|_{H_s^\gamma} + \epsilon^{p-(\frac{\rho}{2}+\frac{1}{4})}\right), \tag{11}$$

and the front evolution is given by (54). In particular, after the remainder W has decayed to $\mathcal{O}(\epsilon^{p-(\frac{\rho}{2}+\frac{1}{4})})$, the pulse evolution is given by

$$y_0'(t) = \sqrt{\epsilon}v(y_0) + \epsilon^{p+\frac{1}{4}-\frac{\rho}{2}} \left(\eta_s, \psi^\dagger(y_0)\right)_{L^2_s} + \mathcal{O}(\epsilon), \tag{12}$$

where the formal pulse velocity v is constructed in Lemma 1 and $\psi^\dagger$ is defined in Lemma 3. In particular for time $t \gg 1/\sqrt{\epsilon}$ the front will oscillate about a fixed point of the formal velocity $v(y_0)$. The width of the ignition set satisfies

$$|E| \leq M\epsilon^{\frac{p-\rho/2-1/4}{3/2-\gamma}}, \tag{13}$$

for any γ satisfying the constraints above.

Remark 1. For large p the third condition on γ becomes incompatible with the first. We may remove the third condition and retain the decay of the remainder and the leading order pulse evolution as described by (12), however the long-time limit of the remainder in (11) and the leading order correction to the pulse evolution (12) take a different form as they arise not from the noise but from the residual.

Remark 2. The large-time limit of the residual, $\epsilon^{p-(\frac{\rho}{2}+\frac{1}{4})}$ tends to $O(1)$ as p approaches its lower bound, and in this scaling the impact of the noise on the front velocity in (12) achieves the same order as the formal velocity v. However in this limit ϵ_0 approaches zero, and the RG approach is not able to handle the case in which the noise generates an $\mathcal{O}(1)$ variation of the solution from the front profile. In addition the third constraint on γ keeps the exponent on ϵ in (13) larger than $\frac{1}{2}$, precluding the ignition set width from being larger than the $\mathcal{O}(\sqrt{\epsilon})$ front width.

We record in the Lemma below a few observations on the front and its velocity in the fast variables.

Lemma 2. *For each bounded interval $\left(z_0^l, z_0^r\right)$ containing z_0, there exists $\alpha_0 > 0$, independent of ϵ, such that*

$$\phi'(z; z_0) > \alpha_0 > 0, \tag{14}$$

for all $z \in \left(z_0^l, z_0^r\right)$. Moreover the residual in the convected frame is given by

$$R = -(v(z_0) - v(\bar{z}_0))\frac{\partial \phi_i}{\partial z_0} + \epsilon \left[\frac{\partial^2 \phi_o}{\partial z^2}\right]. \tag{15}$$

Additionally we have the following bound

$$\|R\|_{L^2_s} + \|R\|_{L^\infty} \leq C\left(\sqrt{\epsilon}\,|z_0 - \bar{z}_0| + \epsilon\right). \tag{16}$$

The velocity is a slowly varying function of z,

$$|v(z_0) - v(\bar{z}_0)| \leq C\sqrt{\epsilon}\,|z_0 - \bar{z}_0|\,. \tag{17}$$

3.1. **The Frechet derivative of F.** In a neighborhood of the composite solution ϕ we decompose the solution $U(z,t)$ of the model equation (1) as

$$U(z,t) = \phi(z; z_0(t)) + W(z,t), \tag{18}$$

where $\|W\|_{H^\gamma_s} \ll 1$, for some $\gamma \in (\frac{1}{2}, 1)$. A key goal is to determine $\gamma, \beta \in (\frac{1}{2}, 1)$ for which the nonlinearity F is Frechet differentiable as a map from $H^\gamma(\mathbb{R}) \to H^{-\beta}(\mathbb{R})$, at the composite solution $\bar{\phi}$. In this case the evolution for U can be written in terms of the decomposed variables z_0 and W as,

$$W_t + \frac{\partial \phi}{\partial z_0} z_0' = F(\phi) + L_{\bar{\phi}}W + (L_\phi - L_{\bar{\phi}})W + \mathcal{N}(W). \tag{19}$$

Here $L_{\bar\phi}$ is the Frechet derivative of $F : H^\gamma \to H^{-\beta}$ at $\bar\phi = \phi(z; \bar z_0)$, and the nonlinear operator $\mathcal{N}$ is given by,

$$\mathcal{N} = F(\bar\phi + W) - F(\bar\phi) - L_{\bar\phi}W.$$

For g in the dual of H^γ and $f \in H^{-\beta}$ it is convenient to introduce the tensor product

$$(f \otimes g)W = \langle W, g\rangle\, f,$$

so that $f \otimes g$ is a rank-one operator with range f. In particular, when f and g are delta functions located at $\bar z_0$

$$(\delta_{\bar z_0} \otimes \delta_{\bar z_0})W \equiv \langle W, \delta_{\bar z_0}\rangle\, \delta_{\bar z_0} = W(\bar z_0)\delta_{\bar z_0}.$$

For a front $\bar\phi$, we define the "ignition set" $E = E(W)$ of the perturbed front $\bar\phi + W$, to be the set of points

$$E = \left\{ z \middle| \bar\phi(z) + W(z) = \sigma_c \right\},$$

and denote the width of the smallest interval which contains it by $|E|$. In the proposition below we show that the linear impact of H^γ perturbations of $\bar\phi$ on F is to shift the ignition point, an affect which is represented by the last term on the right-hand side of (20). A smaller, nonlinear effect of a H^γ perturbation is to break the monotonicity of the solution, smearing the ignition point into an ignition set.

Proposition 1. *Fix $\gamma, \beta \in \left(\frac{1}{2}, 1\right]$. The function F, given by (10) as a map from H^γ to $H^{-\beta}$, is Frechet differentiable at the composite solution $\bar\phi$ with derivative*

$$L_{\bar\phi} = \partial_z^2 + v\partial_z - I + \mu\chi_{[\bar z_0,\infty]} + \mu\frac{\sigma_c}{\bar\phi'(\bar z_0; \bar z_0)}\delta_{\bar z_0} \otimes \delta_{\bar z_0}. \tag{20}$$

Moreover there exists $\omega_0 > 0$ such that for all $W \in H^\gamma$ with $\|W\|_{H^\gamma_s} \le \omega_0$, the nonlinearity $\mathcal{N}(W)$ satisfies

$$\|\mathcal{N}(W)\|_{H_s^{-\beta}} \le c\left(\|W\|_{H^\gamma_s}^{\beta+1/2} + \|W\|_{H^\gamma_s}^{\gamma+1/2} \right), \tag{21}$$

while the ignition set $E = E(W)$ satisfies

$$|E| \le c\|W\|_{H^\gamma_s}^{\frac{1}{3/2-\gamma}}. \tag{22}$$

Proof: With $L_{\bar\phi}$ given by (20) we show that the nonlinearity

$$\begin{aligned} \mathcal{N}(W) \quad &\equiv \quad F(\bar\phi + W) - F(\bar\phi) - L_{\bar\phi}W, \\ &= \quad \mu\left(\sigma(\bar\phi + W) - \sigma(\bar\phi)\right) - \mu\left(\chi_{[\bar z_0,\infty)}W + \frac{\sigma_c}{\bar\phi'(\bar z_0; \bar z_0)}W(\bar z_0)\delta_{\bar z_0}\right), \end{aligned} \tag{23}$$

satisfies (21). Since $\|W\|_{L^\infty} \le c\|W\|_{H^\gamma_s} \le c\omega_0$, the ignition set E is contained an interval $(\bar z_0 - \varpi, \bar z_0 + \varpi)$ for some $\varpi > 0$. For any $z \in E$ we have the relation

$$\bar\phi(z) + W(z) = \sigma_c = \bar\phi(\bar z_0).$$

Applying the Mean Value theorem to $\bar\phi$ on the interval (z, z_0) (w.l.o.g. $z < \bar z_0$), we obtain

$$W(z) = \bar\phi(\bar z_0) - \bar\phi(z) = (\bar z_0 - z)\,\bar\phi'(\xi_z),$$

where $\xi_z \in (z, \bar z_0)$. Solving for z we obtain the expression

$$z = \bar z_0 - \frac{W(z)}{\bar\phi'(\xi)},$$

valid for each $z \in E$. We denote by $l(z)$ the distance of an arbitrary ignition point, $z \in E$ from the frozen front location $\bar{z}_0$, i.e.

$$l(z) \equiv |\bar{z}_0 - z| = \left|\frac{W(z)}{\bar{\phi}'(\xi_z)}\right|. \tag{24}$$

where $\xi_z \in (z, \bar{z}_0)$. From Lemma 2 we know that

$$\min_{z\in(z,\bar{z}_0)} \bar{\phi}'(z) \geq \alpha_0 > 0,$$

where α_0 depends on ω_0 through ϖ. Thus, there exists $C > 0$ independent of z and ϵ such that

$$l(z) \leq C \|W\|_{H_s^\gamma}. \tag{25}$$

The ignition points of the perturbed front $\bar{\phi} + W$ are linearly close to the ignition point $\bar{z}_0$ of $\bar{\phi}$. Denoting the left and right-most points of E by z_l and z_r, we have

$$\bar{\phi}(z_l) + W(z_l) = \sigma_c = \bar{\phi}(z_r) + W(z_r),$$

however the Hilbert space H^γ is continuously embedded in the Hölder space $\mathcal{C}^{0,\gamma-\frac{1}{2}}$ for $\gamma \in (\frac{1}{2}, \frac{3}{2}]$, so that

$$\left|\bar{\phi}(z_l) - \bar{\phi}(z_r)\right| = |W(z_r) - W(z_l)| \leq C\,|z_l - z_r|^{\gamma-1/2}\,\|W\|_{H_s^\gamma}.$$

Conversely from Lemma 2 there exists $\alpha_0 > 0$ independent of ϵ, such that $\phi'(z) > \alpha_0$ for $z \in (z_l, z_r)$, so that

$$\left|\bar{\phi}(z_l) - \bar{\phi}(z_r)\right| = |z_l - z_r|\left|\bar{\phi}'(\xi)\right| \geq \alpha_0\,|z_l - z_r|.$$

These relations together imply

$$\alpha_0\,|z_l - z_r| \leq |z_l - z_r|^{\gamma-1/2}\,\|W\|_{H_s^\gamma},$$

which verifies (22),

$$|z_l - z_r| \leq C\,\|W\|_{H_s^\gamma}^{\frac{1}{3/2-\gamma}} \leq C\|W\|_{H_s^\gamma}^{\gamma+\frac{1}{2}}. \tag{26}$$

Defining the the set of "false negatives," E_-, and the set of "false positives," E_+,

$$\begin{aligned} E_- &= \left\{z\middle|\phi + W \leq \sigma_c\right\}\bigcap\left\{\bar{z}_0 \leq z \leq z_r\right\}, \\ E_+ &= \left\{z\middle|\phi + W \geq \sigma_c\right\}\bigcap\left\{z_l \leq z \leq \bar{z}_0\right\}, \end{aligned}$$

we decompose $\sigma(\bar{\phi} + W)$ in terms of E_+ and E_- as follows

$$\sigma(\bar{\phi} + W) = \left(\chi_{E_+} + \chi_{[\bar{z}_0,\infty)} - \chi_{E_-}\right)(\bar{\phi} + W). \tag{27}$$

For simplicity of presentation we assume that E_- is empty, and address the $H^{-\beta}$ norm of the nonlinearity,

$$\|\mathcal{N}(W)\|_{H_s^{-\beta}} = \sup_{v\in H^\beta} \frac{|\langle\mathcal{N}(W), v\rangle|}{\|v\|_{H_s^\beta}}.$$

As depicted in Fig(2), we denote by $z_l^m \equiv \max\limits_{z\in E_+} z \leq \bar{z}_0$, the maximum element of E_+, so that, $\bar{\phi} + W - \sigma_c > 0$ on (z_l^m, ∞). We partition E_+ as

$$E_+ = E_+^0 \bigcup [z_l^m, \bar{z}_0],$$

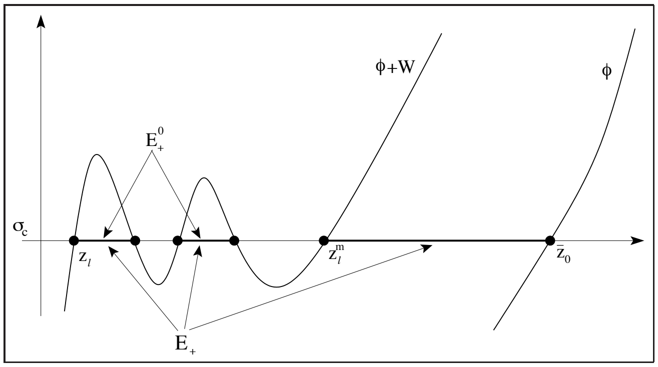

FIGURE 2. The graph of $\bar{\phi} + W$ in the neighborhood of $\bar{z}_0$, in the case that E_- is empty. The ignition set, from z_l to z_l^m is thinner than the distance $|z_l^m - \overline{z}_0|$. The set E_+ consists of the false positives, while E_+^0 is the nonlinear component of the false positives.

where E_+^0 contains the intervals on the left of z_l^m, for which $\bar{\phi} + W - \sigma_c > 0$. Generically the interval $[z_l^m, z_0]$ is the largest share of E_+, and the decomposition (27) reduces to

$$\sigma(\bar{\phi} + W) = (\bar{\phi} + W)(\chi_{E_+^0} + \chi_{[z_l^m,\infty)}). \tag{28}$$

Substituting the decomposition (28) into equation (23), the nonlinearity reduces to

$$\mathcal{N}(W) = \mu \left(\chi_{E_+^0} + \chi_{[z_l^m,\bar{z}_0]}\right)(\bar{\phi} + W) - \mu l_0 \bar{\phi}(\bar{z}_0)\delta_{\bar{z}_0}, \tag{29}$$

where we have introduced the length scale $l_0 = \frac{W(\bar{z}_0)}{\bar{\phi}'(\bar{z}_0)}$, which we will compare to $|z_l^m - \bar{z}_0|$, the dominant part of E_+. The dual pairing which appears in the $H^{-\beta}$ norm of $\mathcal{N}$ takes the form

$$\begin{aligned} |\langle \mathcal{N}(W), v\rangle| \quad &\leq \quad \mu \left| \int_{E_+^0} \left(\bar{\phi} + W\right) v dz \right| + \mu \left| \int_{z_l^m}^{\bar{z}_0} \bar{\phi} v - \bar{\phi}(\bar{z}_0) v(\bar{z}_0) dz \right| \\ &+ \quad \mu \left| l(z_l^m) - l_0 \right| \left|\bar{\phi}(\bar{z}_0)\right| |v(\bar{z}_0)| + \mu \left| \int_{z_l^m}^{\bar{z}_0} W v dz \right|, \end{aligned} \tag{30}$$

where $l(z_l^m)$ is defined in (24). Using (26) we have the bound

$$\left| \int_{E_+^0} \left(\bar{\phi} + W\right) v dz \right| \leq C|E_+^0| \, \|v\|_{L^\infty} \leq C \, \|W\|_{H_s^\gamma}^{\gamma+1/2} \, \|v\|_{H_s^\beta}. \tag{31}$$

To investigate the second term, we add and subtract $v(\bar{z}_0)\bar{\phi}$

$$\left| \int_{z_l^m}^{\bar{z}_0} \bar{\phi} v - \bar{\phi}(\bar{z}_0) v(\bar{z}_0) dz \right| \leq \left| \int_{z_l^m}^{\bar{z}_0} \bar{\phi}\left(v - v(\bar{z}_0)\right) dz \right| + \left| \int_{z_l^m}^{\bar{z}_0} v(\bar{z}_0) \left(\bar{\phi} - \bar{\phi}(\bar{z}_0)\right) dz \right|.$$

However $|v(z)-v(z_0)|\leq c|z-z_0|^{\beta-\frac{1}{2}}\|v\|_{H^\beta_s}$ so that using (25) we obtain

$$\left|\int_{\bar z_0}^{z_l^m}\bar\phi v-\bar\phi(\bar z_0)v(\bar z_0)dz\right|\leq C\|v\|_{H^\gamma_s}\int_{\bar z_0}^{z_l^m}|z-\bar z_0|^{\beta-1/2}\,dz\leq C\,\|W\|^{\beta+1/2}_{H^\gamma_s}\,\|v\|_{H^\beta_s}. \quad (32)$$

The third term on the right hand side of (30), balances the dominant component of E_+ against the rank-one tensor product of the delta functions within the linear operator. We recall that $\bar\phi(\bar z_0)=\sigma_c$, hence

$$|l(z_l^m)-l_0|\,\left|\bar\phi(\bar z_0)\right|\,|v(\bar z_0)|\leq\sigma_c\,\|v\|_{H^\beta_s}\,|l(z_l^m)-l_0|$$

To bound $|l(z_l^m)-l_0|$ we recall the definitions of $l(z_l^m)$ and l_0,

$$|l(z_l^m)-l_0|=\left|\frac{W(z_l^m)}{\bar\phi'(\xi)}-\frac{W(\bar z_0)}{\bar\phi'(\bar z_0)}\right|=\frac{\left|W(z_l^m)\bar\phi'(\bar z_0)-W(\bar z_0)\bar\phi'(\xi)\right|}{\left|\bar\phi'(\xi)\bar\phi'(\bar z_0)\right|}.$$

where $\xi\in(z_l^m,\bar z_0)$. Since $\bar\phi'(\xi)$ and $\bar\phi'(\bar z_0)$ are uniformly bounded below and $W\in\mathcal{C}^{0,\gamma-\frac{1}{2}}$ we have the bound

$$|l(z_l^m)-l_0|\leq C\,|z_l^m-\bar z_0|^{\gamma-1/2}\,\|W\|_{H^\gamma_s}\leq c\,\|W\|^{\gamma+1/2}_{H^\gamma_s}. \quad (33)$$

For the fourth term, from (25) we estimate

$$\left|\int_{z_l^m}^{\bar z_0}Wvdz\right|\leq C\,\|v\|_{H^\beta_s}\,\|W\|_{H^\gamma_s}\,|\bar z_0-z_l^m|\leq c\,\|W\|^2_{H^\gamma_s}\,\|v\|_{H^\beta_s}. \quad (34)$$

The bounds (31), (32), (33) and (34), inserted into (30) yield the desired result (21). □

3.2. **Spectrum of the linearized operator.** It follows from classical results, see Theorem A.2 of [4] for example, that the essential spectrum of the linearized operator $L_{\bar\phi}$, given in (20), lies inside or on the curves $S_\pm$ given by,

$$\begin{aligned}S_- &= \left\{-k^2-vik-1\middle|k\in\mathbb{R}\right\},\\ S_+ &= \left\{-k^2-vik-1+\mu\middle|k\in\mathbb{R}\right\}.\end{aligned}$$

The point spectrum, $\Sigma_{\rm pt}(L_{\bar\phi})$, consists of the broken translational eigenvalue, as seen in the following Lemma.

Lemma 3. *The point spectrum of $L_{\bar\phi}$ is real and satisfies $\Sigma_{pt}(L_{\bar\phi})\bigcap[-1+\mu,\infty)=\{\lambda_p\}$, where λ_p is simple, negative and is given by (36). The associated eigenfunction ψ and adjoint eigenfunction $\psi^\dagger$ belong to H^γ for any $\gamma<\frac{3}{2}$, and after normalization satisfy*

$$\left\|\psi-\frac{\partial\phi}{\partial\bar z_0}\right\|_{L^2_s}\leq C\sqrt{\epsilon}, \quad (35)$$

where $C>0$ is independent of $\bar z_0$.

Proof: We outline the proof, for details see [17]. The eigenfunctions corresponding to the point spectrum of $L_{\bar\phi}$ satisfy

$$\begin{aligned}\psi''+v\psi'-(\lambda_p+1)\psi+\mu\chi_[\bar z_0,\infty)\psi=0.\\ [\![\psi']\!]_{\bar z_0}=-\frac{\sigma_c}{\bar\phi'(\bar z_0)}\psi(\bar z_0).\end{aligned}$$

It is straightforward to verify that the point eigenfunctions must decay at least as fast as $\mathcal{O}(e^{-\bar{v}|\xi-\bar{z}_0|/2})$, as $\xi \to \pm\infty$. The operator $L_{\bar{\phi}}$ may then be verified to be self-adjoint in the weighted inner product

$$\langle u, v\rangle_\nu = \int_{\mathbb{R}} e^{\bar{v}(z-\bar{z}_0)} u(z)v(z)dz,$$

which verifies that the spectrum is real. The eigenvalue problem is piece-wise constant coefficient with a jump condition, so that any eigenfunction has the form

$$\psi(z) = \begin{cases} \frac{1}{\theta_1} e^{a(z-\bar{z}_0)} & \xi \leq \bar{z}_0, \\ \frac{1}{\theta_1} e^{b(z-\bar{z}_0)} & \xi \geq \bar{z}_0, \end{cases}$$

where $a = \frac{1}{2}(-v + \sqrt{v^2 + 4(1+\lambda_p)})$, $b = -\frac{1}{2}(v + \sqrt{v^2 + 4(1+\lambda_p - \mu)}$, and θ_1 is a normalization. The jump condition on the derivative of ψ requires that the eigenvalue λ_p satisfy the equation

$$\sqrt{v^2 + 4(1+\lambda_p - \mu)} + \sqrt{v^2 + 4(1+\lambda_p)} = \frac{2\mu\sigma_c}{\bar{\phi}'(\bar{z}_0)},$$

and since the left-hand side is a strictly increasing function of λ_p on $[-1+\mu, \infty)$, there is exactly one solution. An asymptotic analysis shows that

$$\lambda_p = -\sqrt{\epsilon} \frac{4\mu\sigma_c \sqrt{v^2 + 4(1-\mu)}\sqrt{v^2+4}}{(\sigma_c - g_s(\bar{z}_0))(\sqrt{v^2 + 4(1-\mu)} + \sqrt{v^2+4})} \frac{dg(\bar{y}_0)}{dy}. \tag{36}$$

Since the eigenfunctions are piecewise smooth with a jump in their derivative they are in H^γ for any $\gamma < \frac{3}{2}$.

Since $\lambda_p = \mathcal{O}(\sqrt{\epsilon})$ we see that $\lambda_1^+ = a + \mathcal{O}(\sqrt{\epsilon})$ and $\lambda_2^- = b + \mathcal{O}(\sqrt{\epsilon})$. Moreover v and g vary slowly with $\bar{z}_0$; taking the $\partial_{\bar{z}_0}$ of (7) shows that $\partial_{\bar{z}_0}\phi$ is, to $O(\sqrt{\epsilon})$, a continuous function comprised piecewise of exponentials which decay away from $z = \bar{z}_0$ at the same leading order rate as ϕ. As such functions comprise a one dimensional space the two functions agree, after appropriate normalization, to leading order. □

4. **Resolvent and semigroup estimates.** To determine the behavior of the semigroup, $S_{\bar{z}_0}$, associated to $L_{\bar{\phi}}$, we must first we study the resolvent operator, $(L_{\bar{\phi}} - \lambda)^{-1} : H^{-\beta} \to H^\gamma$ for $\gamma, \beta \in \left(\frac{1}{2}, 1\right]$. We introduce the spectral projection $\pi_{\bar{z}_0}$,

$$\pi_{\bar{z}_0} U = (U, \psi^\dagger_{\bar{z}_0})_{L^2_s} \psi_{\bar{z}_0}, \tag{37}$$

and its compliment $\tilde{\pi}_0 = I - \pi_0$, where the eigenfunctions have been scaled so that $(\psi_{\bar{z}_0}, \psi^\dagger_{zb})_{L^2_s} = 1$. The associated eigenspace $X_{\bar{z}_0} = \left\{U \middle| \|U\|_{H^\gamma_s} < \infty \text{ and } \pi_{\bar{z}_0} U = 0\right\}$, corresponds to the $L_{\bar{z}_0}$ spectral set which is uniformly in the left-half complex plane.

We fix the contour $\mathcal{C}$ in $\rho(L_{\bar{\phi}})$ as depicted in Figure 3 where the parameter l shifts the contour horizontally, m determines the opening angle of the branches, and $a = 1 + v^2/4$.

4.1. **Decomposition of the linearized operator.** To simplify notation we introduce $\alpha = \mu \frac{\sigma_c}{\bar{\phi}'(\bar{z}_0)} > 0$ and decompose L as

$$L = \mathcal{L} + \alpha \delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0} + \mu \chi_{(\bar{z}_0, \infty)}, \tag{38}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_z^2 + v\partial_z - I.$$

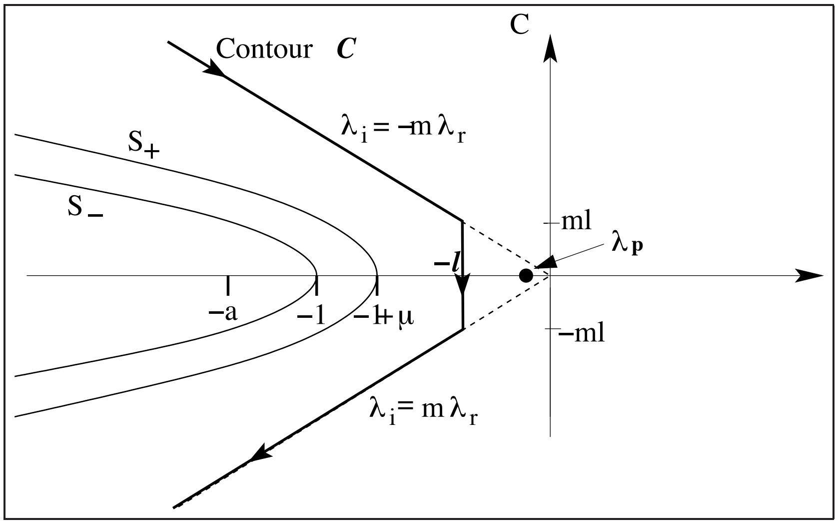

FIGURE 3. The contour $\mathcal{C}$, the essential spectrum and the point spectrum of L.

Lemma 4. *Fix $\gamma+\beta<2$. There exists $c>0$ such that for all λ on the contour $\mathcal{C}$ and $F\in H_s^{-\beta}$ the following estimate holds*

$$\left\|(\mathcal{L}-\lambda)^{-1}F\right\|_{H_s^\gamma} \le c(|\lambda+a|)^{(\gamma+\beta-2)/2}\,\|F\|_{H_s^{-\beta}}\,. \tag{39}$$

Proof: Taking the Fourier transform of $(\mathcal{L}-\lambda)u=F$ we obtain,

$$\hat{u}(k)=\frac{\hat{F}(k)}{-k^2-ikv-(1+\lambda)}.$$

Taking the H_s^γ norm of u and applying Plancherel we find

$$\|u\|_{H_s^\gamma}^2 \le \|G\|_{L^\infty}\,\|F\|_{H_s^{-\beta}}^2\,,$$

where,

$$G(k)=\frac{(1+k^2)^{\gamma+\beta}}{\left|k^2+ikv+1+\lambda\right|^2}.$$

We observe that

$$\left|k^2+ikv+1+\lambda\right| \ge \mathrm{dist}(\lambda,-S_-),$$

so the denominator is bounded from below. For $|\lambda|\gg 1$ it is easy to verify that

$$\|G\|_{L^\infty} \le \frac{C}{|\lambda+a|^{2-\gamma-\beta}},$$

which yields (39). □

To regularize the resolvent equation we write

$$u\equiv (L-\lambda)^{-1}F=(\mathcal{L}-\lambda)^{-1}F+w,$$

where w solves

$$(L-\lambda)w=F-(L-\lambda)(\mathcal{L}-\lambda)^{-1}F=\kappa\delta_{\bar{z}_0}+G. \tag{40}$$

Here, for notational convenience we have defined $\kappa \equiv -\alpha \left((\mathcal{L}-\lambda)^{-1}F\right)(\bar{z}_0) \in \mathbb{R}$ and $G \equiv -\mu \left((\mathcal{L}-\lambda)^{-1}F\right)\chi_{(\bar{z}_0,\infty)} \in L^2(\mathbb{R}) \cap H^\gamma(\bar{z}_0,\infty)$.

Lemma 5. *Let $\gamma, \beta \in (\frac{1}{2}, 1)$ satisfy $\gamma + \beta < 2$, and let m be sufficiently large, then $\exists C > 0$, independent of $\epsilon > 0$, such that for all λ on the contour $\mathcal{C}$ and $F \in H^{-\beta}$, the solution w of (40) satisfies*

$$\|w\|_{H_s^\gamma} \leq C\,|\lambda + a|^{(\gamma+\beta-2)/2}\,\|F\|_{H_s^{-\beta}}\,. \tag{41}$$

Proof: The equation (40) for w can be written as

$$\begin{aligned}
w'' + vw' - (1+\lambda)w &= 0 \quad \text{for} \quad z < \bar{z}_0,\\
w'' + vw' - (1+\lambda-\mu)w &= G \quad \text{for} \quad z > \bar{z}_0,\\
[\![w]\!]_{z=\bar{z}_0} &= 0,\\
[\![w']\!]_{z=\bar{z}_0} &= \kappa + \alpha w(\bar{z}_0).
\end{aligned}$$

For $z < \bar{z}_0$, the solutions take the form

$$w(z) = A_1 e^{\nu_1^+(z-\bar{z}_0)} + B_1 e^{\nu_1^-(z-\bar{z}_0)},$$

where $\nu_1^\pm = \frac{1}{2}(-v \pm \sqrt{v^2+4+4\lambda})$. There exists $\nu > 0$ such that $\mathrm{Re}(\nu_1^+) > \nu > -\nu > \mathrm{Re}(\nu_1^-)$ so long as $-a \leq -l \leq \frac{-4a}{m^2}$. For $z > \bar{z}_0$, we the inhomogeneous equation has solutions

$$w(z) = A_2 e^{\nu_2^+(z-\bar{z}_0)} + B_2 e^{\nu_2^-(z-\bar{z}_0)} + \int_{\bar{z}_0}^{z} R(z,\xi)G(\xi)d\xi,$$

where $R(z,\xi)$ is defined as

$$R(z,\xi) = \frac{e^{\nu_2^-(z-\xi)} - e^{\nu_2^+(z-\xi)}}{\nu_2^- - \nu_2^+},$$

and $\nu_2^\pm = \frac{1}{2}(-v \pm \sqrt{v^2+4(1+\lambda-\mu)})$. For $\lambda \in \mathcal{C}$ there exists $\nu' > 0$ such that $\mathrm{Re}(\nu_2^+) \geq \nu' \geq -\nu' \geq \mathrm{Re}(\nu_2^-)$ so long as $-a+\mu \leq -l \leq \frac{-4a(1-\mu)}{m^2}$. Both conditions on the contour $\mathcal{C}$ may be satisfied if $\frac{4a}{m^2} < l < a-\mu$, which is feasible so long as $m^2 > \frac{4a}{a-\mu}$. Under these conditions the operator L has an exponential dichotomy for λ on the contour $\mathcal{C}$ and the result (41) follows from standard results, see for instance Lemma 1 of Appendix to chapter 5 of [4]. □

The semigroup $S_{\bar{\phi}}(t) \equiv S(t)$ generated by $L_{\bar{\phi}}$ as a map from $H^{-\beta}$ to H^γ can be represented as the Laplace transform of the resolvent,

$$S(t)\tilde{\pi}_0 F = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{\lambda t}(L-\lambda)^{-1}F d\lambda, \tag{42}$$

for $F \in H^{-\beta}$, where $\tilde{\pi}_0$ is the spectral projection off of the eigenspace associated to the eigenvalue λ_p, which is outside the contour. The following Lemma bounds the resolvent and the semigroup.

Lemma 6. *Let γ and β satisfy $\gamma + \beta < 2$ and let $m > 0$ be sufficiently large, then $\exists C > 0$ such that for all λ on the contour $\mathcal{C}$ and $F \in H^{-\beta}$*

$$\left\|(L-\lambda)^{-1}F\right\|_{H_s^\gamma} \leq \frac{C\,\|F\|_{H_s^{-\beta}}}{(|\lambda+a|)^{(2-\gamma-\beta)/2}}. \tag{43}$$

Moreover the semigroup satisfies the bound

$$\|S(t)\tilde{\pi}_0 F\|_{H_s^\gamma} \leq \frac{Ce^{-\nu t}}{t_<^{(\gamma+\beta)/2}} \|F\|_{H_s^{-\beta}}, \tag{44}$$

for all $F \in H^{-\beta}$*, where* $t_< = \min\{t, 1\}$.

Proof: Let $u = (L-\lambda)^{-1}F$, from the decomposition (40) and Lemmas 4 and 5 we find that

$$\|u\|_{H_s^\gamma} \leq c\|(\mathcal{L}-\lambda)^{-1}F\|_{H_s^\gamma} + \|w\|_{H_s^\gamma} \leq C|\lambda+a|^{(\gamma+\beta-2)/2}\|F\|_{H_s^{-\beta}},$$

which establishes (43). Writing $\lambda = \lambda_r + i\lambda_i$, for λ on the contour $\mathcal{C}$, and using (43) we have the estimate

$$\|S(t)\tilde{\pi}_0 F\|_{H_s^\gamma} \leq C \int\limits_{\mathcal{C}} e^{\lambda_r t} \frac{\|\tilde{\pi}_0 F\|_{H_s^{-\beta}}}{|\lambda+a|^{(2-\gamma-\beta)/2}} d\lambda.$$

For λ on the contour, there exists constants $\nu > 0$ and $\omega > 0$ such that, $\lambda_r \leq -(\nu + \omega\,|\lambda|)$. Since ψ and $\psi^\dagger$ lie in H^1, the projection $\tilde{\pi}_0$ is continuous on $H_s^{-\beta}$ and we obtain

$$\|S(t)\tilde{\pi}_0 F\|_{H_s^\gamma} \leq Ce^{-\nu t}\,\|F\|_{H_s^{-\beta}} \int\limits_{\mathcal{C}} \frac{e^{\omega\lambda_r t}}{|\lambda+a|^{(2-\gamma-\beta)/2}} d\lambda.$$

For large time the integral is uniformly bounded. For $t \ll 1$ we set $\tilde{\lambda} = \lambda t$ and the integral becomes

$$\|S(t)\tilde{\pi}_0 F\|_{H_s^\gamma} \leq \frac{Ce^{-\nu t}\|F\|_{H_s^{-\beta}}}{t^{(\gamma+\beta)/2}} \int\limits_{t\mathcal{C}} \frac{e^{\omega\tilde{\lambda}_r}}{\left|at^{(2-\gamma-\beta)/2} + \tilde{\lambda}^{(2-\gamma-\beta)/2}\right|} d\tilde{\lambda}.$$

As $t \to 0^+$ the contour $t\mathcal{C}$ converges to the cone $|\tilde{\lambda}_i| = -m\tilde{\lambda}_r$. For $\gamma+\beta < 2$, the integral is uniformly bounded on this limiting contour since the singularity in $\tilde{\lambda}$ at $t = 0$ is integrable, and we obtain the estimate (44). $\square$

5. **The RG methodology.** The RG methodology was developed in [14] for pulse interaction in regular parabolic and hyperbolic systems. It was extended in [1, 15, 22, 23] to singularly perturbed systems with a slow-fast structure. We adapt the method to our nonsmooth system (1), emphasizing the modifications required. The composite solution constructed in Lemma 1 forms a one dimensional manifold

$$\mathcal{M} = \{\phi(\cdot; y_0)\big| y_0 \in \mathcal{K}\},$$

parameterized by the front location. The manifold is not invariant under the dynamics of (1), even in the noiseless setting, however Lemma 3 shows that the linearization, $L_{\bar{\phi}}$, about a particular point, $\bar{\phi}$, on the manifold has a single small eigenvalue, the broken translational invariant ψ, with the remainder of the spectra uniformly in the left-half complex plane. Moreover from (15) and (35) the residual, $R = F(\phi)$ lies at leading order within the tangent plane of the manifold. The perturbations arising from the residual and the noise serve to push the front solution $\phi(\cdot; y_0)$ along the manifold, while their ability to pull it off the manifold is mitigated by the uniform damping in that direction.

In the RG approach we temporarily freeze the coordinate system, including the linearized operator, $\bar{\phi}$, the front velocity, $\bar{v}$, and the spectral subspace, $\tilde{X}$, and solve a series of initial value problems on disjoint intervals $I_n = [t_n, t_{n+1}]$ for $n = 0, 1, \ldots,$

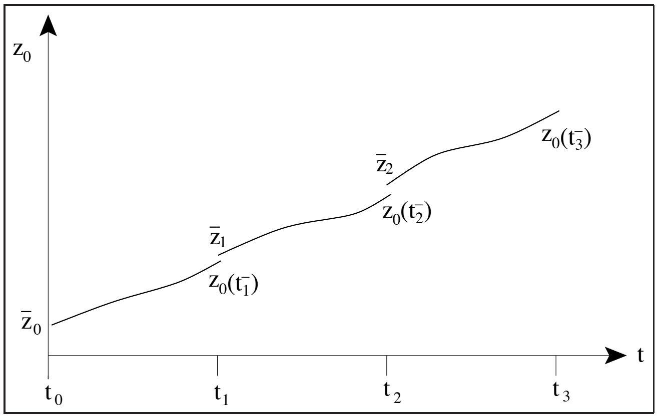

FIGURE 4. Evolution of front location, including jumps at renormalization.

which are specified dynamically as the solution evolves. Given the solution U at time t_n, the frozen front location $\bar{z}_n$ is determined so that

$$W_n(0) \equiv U(\cdot, t_n) - \phi(z; \bar{z}_n) \in \tilde{X}_{\bar{z}_n}. \tag{45}$$

For $t \in I_n$, we impose the nonsecularity condition $W, W_t \in \tilde{X}_n$ for $t \in I_n$, and project the evolution for the remainder, (19), onto $\tilde{X}_n$, obtaining the initial value problem

$$W_t = L_{\bar{z}_n} W + \tilde{R} + \tilde{\pi}_n \left(\Delta L\, W + \mathcal{N}(W) + \epsilon^{p_s} \eta_s \right), \tag{46}$$

$$W(t_n) = W_n, \tag{47}$$

where the secular term $\Delta L = L_{z_0} - L_{\bar{z}_n}$ arises from freezing the linear operator, and the quasi-static residual $\tilde{R}$ is given by

$$\tilde{R} = \tilde{\pi}_n \left(R - z_0' \frac{\partial \phi}{\partial z_0} \right).$$

The nonsecularity condition determines the front evolution, obtained by acting on the evolution equation (19) with π_n,

$$\pi_n \frac{\partial \phi}{\partial z_0} z_0' = \pi_n \left(F(\phi) + \Delta L\, W + \mathcal{N}(W) + \epsilon^{p_s} \eta_s \right), \tag{48}$$

$$z_0(t_n) = \bar{z}_n. \tag{49}$$

As z_0 evolves away from $\bar{z}_n$ the secular term, ΔL, grows and control of the remainder is lost when the secular term reaches a critical size, signaling the end of the interval I_n. As depicted in Figure 4, the front location z_0 is discontinuous at each renormalization time, jumping from $z_0(t_n^-)$ to $\bar{z}_n$ at $t = t_n$. The structure of the initial value problems is upscaled into the RG equation

$$\mathcal{G}(W_n, \bar{z}_n) \equiv (W_{n+1}, \bar{z}_{n+1}), \tag{50}$$

which captures the slow dynamics of the system a map from initial data to initial data.

Throughout this section a subscript $\bar{z}_n$ indicates a quantity associated to the coordinate system centered about the base point $z = \bar{z}_n$. For example, $\psi_{\bar{z}_n} = \psi(z; \bar{z}_n)$

denotes the principle eigenfunction of $L_{\bar{z}_n}$ where $L_{\bar{z}_n}$ denotes the linearization of F at $\phi(z;\bar{z}_n)$.

We assume at time $t = t_0$, the initial data U_0 satisfies,

$$\left\|\phi_{z_0^*} - U_0\right\|_{H_s^\gamma} \leq \delta,$$

for some $z_0^* \in \mathcal{K}$, where $\delta > 0$ will be specified later. The following proposition details the solution of the projection equation (45) which extracts the new coordinate system form the expired one.

Proposition 2. *Fix $\gamma \in (\frac{1}{2}, 1)$ and let $\delta > 0$ be sufficiently small. Assume $U_0 \in H_s^\gamma$ and $z_* \in \mathcal{K}$ satisfy $\|W_*\|_{H_s^\gamma} \leq \delta$, where the remainder $W_* \equiv U_0 - \phi_{z_*}$. There exists $M > 0$, independent of U_0 and z_*, and a smooth function $\mathcal{H} : H_s^\gamma \to \mathbb{R}$ such that $\bar{z}_0 = z_* + \mathcal{H}(W_*)$ satisfies,*

$$W_0 \equiv U_0 - \phi_{\bar{z}_0} \in X_{\bar{z}_0}. \tag{51}$$

Moreover, if $W_* \in X_{\tilde{z}_*}$ for some $\tilde{z}_* \in \mathcal{K}$ then

$$\|W_0 - W_*\|_{H_s^\gamma} + |\bar{z}_0 - z_*| \leq M_0 \left|z_* - \tilde{z}_*\right| \|W_*\|_{H_s^\gamma}. \tag{52}$$

Proof: Since $W_0 = U_0 - \phi_{\bar{z}_0} = W_* + \phi_{z_*} - \phi_{\bar{z}_0}$, constructing $\mathcal{H}$ requires solving the equation

$$\Lambda(\bar{z}_0, W_*) \equiv \left(W_* + \phi_{z_*} - \phi_{\bar{z}_0}, \psi_{\bar{z}_0}^\dagger\right)_{L_s^2} = 0.$$

We observe that $\Lambda(z_*, 0) = 0$ and from (35) we have

$$\frac{\partial \Lambda}{\partial \bar{z}_0}\Big|_{(z_*,0)} = \left(\frac{\partial \phi_{z_0}}{\partial z_0}, \psi_{\bar{z}_0}^\dagger\right)_{L_s^2}\Big|_{(z_*,0)} = \left(\psi_{\bar{z}_0}, \psi_{\bar{z}_0}^\dagger\right)_{L_s^2} + \mathcal{O}(\sqrt{\epsilon}) = 1 + \mathcal{O}(\sqrt{\epsilon}) > 0,$$

and the implicit function theorem guarantees the existence of a $\mathcal{C}^1$ function $\mathcal{H}$ for which $\Lambda(\mathcal{H}(W_*), W_*) = 0$ in a neighborhood of the manifold $\mathcal{M}$. The size of this neighborhood determines δ.

If in addition $W_* \in X_{\tilde{z}_*}$ then $\left(W_*, \psi_{\tilde{z}_*}^\dagger\right)_{L_s^2} = 0$ and

$$\left|\left(W_*, \psi_{z_*}^\dagger\right)\right| = \left|\left(W_*, \psi_{z_*}^\dagger - \psi_{\tilde{z}_*}^\dagger\right)\right| \leq c\|W_*\|_{H_s^\gamma} |z_* - \tilde{z}_*|,$$

since $\psi^\dagger$ is $\mathcal{C}^1$. The result (52) follows since $|\partial_{z_0}\Lambda|$ is uniformly bounded away from zero and $W_0 - W_* = \phi_{\bar{z}_0} - \phi_{z_0^*}$. ◻

5.1. **Control of the remainder.** With Proposition 2 in hand we may define the initial data, which we denote W_0 and $\bar{z}_0$, for the initial value problems (46)-(49). From the semigroup estimates (44) we anticipate that $\|W\|_{H_s^\gamma}$ will decay exponentially with rate $\nu > 0$. To control the front dynamics we introduce the quantities,

$$\begin{aligned} T_0(\tau) &= \sup_{t_0 < s < t_0 + \tau} e^{\nu(s - t_0)} \|W(s)\|_{H_s^\gamma}, \\ T_1(\tau) &= \sup_{t_0 < s < t_0 + \tau} |z_0(s) - \bar{z}_0|. \end{aligned}$$

We control the secular term with the following Lemma.

Lemma 7. *Fix $\gamma \in (\frac{1}{2}, 1)$, then there exists $c > 0$ such that for all $\bar{z}_0, z_0 \in \mathcal{K}$ and $W \in H_s^\gamma$ we have the estimate*

$$\|\Delta L\, W\|_{H_s^{-\gamma}} \leq c T_1^{\gamma - \frac{1}{2}} \|W\|_{H_s^\gamma}. \tag{53}$$

Proof: From the definition, (20), of the linear operator L, we see that the secular term takes the form

$$\Delta L\, W = (v(z_0) - v(\bar{z}_0))\partial_z W + \sigma_c \mu \left(\frac{W(z_0)}{\phi'(z_0; z_0)} \delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0; \bar{z}_0)} \delta_{\bar{z}_0} \right) + \mu \chi_{(\bar{z}_0, z_0)} W.$$

From (17) we see

$$\|(v(z_0) - v(\bar{z}_0))\partial_z W\|_{H_s^{-\gamma}} \le c\sqrt{\epsilon} T_1 \|W\|_{H_s^{1-\gamma}} \le c\sqrt{\epsilon} T_1 \|W\|_{H_s^{\gamma}},$$

since $\gamma > \frac{1}{2}$. Recalling the definition of $H_s^{-\gamma}$ norm

$$\left\| \frac{W(z_0)}{\phi'(z_0)} \delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} \delta_{\bar{z}_0} \right\|_{H_s^{-\gamma}} = \sup_{\|v\|_{H_s^{\gamma}} = 1} \left| \frac{W(z_0)}{\phi'(z_0)} v(z_0) - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} v(\bar{z}_0) \right|.$$

By Lemma 2 ϕ' is uniformly bounded away from zero on $(z_0, \bar{z}_0)$, and moreover $W, v \in \mathcal{C}^{0,\gamma-\frac{1}{2}}$ are both holder continuous so by adding and subtracting terms we see

$$\left\| \frac{W(z_0)}{\phi'(z_0)} \delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} \delta_{\bar{z}_0} \right\|_{H_s^{-\gamma}} \le c T_1^{\gamma - 1/2} \|W\|_{H_s^{\gamma}}.$$

Finally the third term in the secularity satisfies

$$\|\chi_{(\bar{z}_0, z_0)} W\|_{H_s^{-\gamma}} \le c T_1 \|W\|_{L^\infty} \le c T_1 \|W\|_{H_s^{\gamma}}.$$

Combining the estimates on the three terms of the secularity yields (53). □

We first seek to estimate the growth of T_1 which controls the secular terms. Using equation (35) we see that $\left(\frac{\partial \phi}{\partial z_0}(z_0), \phi_{\bar{z}_0} \right)_{L_s^2} = 1 + \mathcal{O}(\sqrt{\epsilon}, T_1)$, and solving (48) for z_0' we obtain the front evolution equation,

$$z_0' = \frac{\left(R + \Delta L\, W + \mathcal{N}(W) + \epsilon^{p_s} \eta_s(\cdot, t), \psi_{\bar{z}_0}^{\dagger} \right)_{L_s^2}}{1 + \mathcal{O}(\sqrt{\epsilon}, T_1)}. \tag{54}$$

Estimating the residual with (16), the secular term with (53), the nonlinearity with (21), and the noise term with Lemma (3), we have the bound

$$\begin{aligned} |z_0'(t)| &\le c \left(\sqrt{\epsilon} T_1(t) + \epsilon + T_1^{\gamma - \frac{1}{2}}(t) \|W(t)\|_{H_s^{\gamma}} + \|W(t)\|_{H_s^{\gamma}}^{\gamma + \frac{1}{2}} + \epsilon^{p_s} \right), \\ &\le c \left(T_1^{\gamma + \frac{1}{2}} + \|W\|_{H_s^{\gamma}}^{\gamma + \frac{1}{2}} + \epsilon^{p_s} \right), \end{aligned}$$

where we used Young's inequality and the fact that $p_s < 1$ to obtain the final expression on the second line. Integrating z_0' over the interval $(t_0, t_0 + \tau)$, we find

$$\begin{aligned} T_1(\tau) &\le c \int_{t_0}^{t_0 + \tau} \left(T_1^{\gamma + \frac{1}{2}}(\tau) + e^{-(\gamma + \frac{1}{2})\nu(s - t_0)} T_0^{\gamma + \frac{1}{2}}(\tau) + \epsilon^{p_s} \right) ds, \\ &\le c \left(\tau T_1^{\gamma + \frac{1}{2}} + T_0^{\gamma + \frac{1}{2}} + \tau \epsilon^{p_s} \right). \end{aligned}$$

So long as the constraint

$$c \tau T_1^{\gamma - \frac{1}{2}} \le \frac{1}{2}, \tag{55}$$

holds we may neglect the T_1 term on the right-hand side and obtain the bound

$$T_1 \le c \left(T_0^{\gamma + \frac{1}{2}} + \tau \epsilon^{p_s} \right). \tag{56}$$

The following Lemma bounds the quasi-static residual.

Lemma 8. *Fix $\gamma \in (\frac{1}{2}, 1)$. On each interval I_n on which the constraint (55) holds, the quasi-static residual, introduced in (46), satisfies the following bound,*

$$\|\tilde{R}\|_{L^2_s} \le c\left((\tau\epsilon^{p_s})^{\gamma+\frac{3}{2}} + T_0^{\gamma+\frac{3}{2}}\right). \tag{57}$$

Proof: From the asymptotic result (35) and the form of the spectral projection, (37), it follows that

$$\left\|\tilde{\pi}_n \frac{\partial \phi}{\partial z_0}(\cdot; z_0)\right\|_{L^2_s} \le c(\sqrt{\epsilon} + T_1).$$

From the form of the residual, (15), and the slow variation of the velocity with z_0, (17), we find

$$\|\tilde{\pi}_n R\|_{L^2_s} \le c\left(\epsilon + \sqrt{\epsilon} T_1 \left\|\tilde{\pi}_n \frac{\partial \phi}{\partial z_0}\right\|_{L^2_s}\right) \le c\left(\epsilon + \sqrt{\epsilon} T_1^2\right).$$

Similarly

$$\left\|\tilde{\pi}_n z_0' \frac{\partial \phi}{\partial z_0}\right\|_{L^2_s} \le c|z_0'|\left(\sqrt{\epsilon} + T_1\right) \le c\left(T_1^{\gamma+\frac{1}{2}} + T_0^{\gamma+\frac{1}{2}} + \epsilon^{p_s}\right)\left(\sqrt{\epsilon} + T_1\right).$$

Combining these two bounds, using Young's inequality, and observing that $2\gamma + 1 > \gamma + \frac{3}{2}$, we obtain

$$\|\tilde{R}\|_{L^2_s} \le c\left(\epsilon + T_1^{\gamma+\frac{3}{2}} + T_0^{\gamma+\frac{3}{2}}\right).$$

The estimate (56), Young's inequality, and the bound $p_s(\gamma + \frac{3}{2}) < 1$ arising from the noise dominant scaling yields the result (57). □

Returning to the evolution equation (46) we apply the frozen coefficient semigroup $S = S(\tau; \bar{z}_0)$ associated to $L_{\bar{z}_0}$, to obtain the variation of constants formula for W,

$$W(t_0 + \tau) = S(\tau)W_0 + \int_{t_0}^{t_0+\tau} S(t_0 + \tau - s)\left(\tilde{R} + \tilde{\pi}_0\left(\Delta L\, W + \mathcal{N}(W) + \epsilon^{p_s}\eta_s(s)\right)\right) ds.$$

Taking the H^γ_s norm of the equation above and applying the semigroup estimate (44) in conjunction with the bounds (53) on the secular term and (21) on the nonlinearity we obtain,

$$\begin{aligned}\|W(t_0 + \tau)\|_{H^\gamma_s} \quad &\le \quad ce^{-\nu\tau}\|W_0\|_{H^\gamma_s} + c\int_{t_0}^{t_0+\tau} e^{-\nu(t_0+\tau-s)}\left(\frac{\|\tilde{R}(s)\|_{L^2_s}}{(s-t_0)_<^{\gamma/2}} + \right. \\ &\left. \frac{T_1^{\gamma-\frac{1}{2}}\|W(s)\|_{H^\gamma_s} + \|W(s)\|_{H^\gamma_s}^{\gamma+\frac{1}{2}}}{(s-t_0)_<^{\gamma}} + \frac{\epsilon^{p_s}}{(s-t_0)_<^{(\gamma+\rho)/2}}\right) ds.\end{aligned}$$

Multiplying the equation above by $e^{\nu\tau}$, taking the supremum over $\tau \in (0, \tau)$, and using the bounds (57) and $\|W(s)\|_{H^\gamma_s} \le e^{-\nu(s-t_0)}T_0$ we find

$$\begin{aligned}T_0(\tau) \quad &\le \quad c\left(T_0(t_0) + \int_{t_0}^{t_0+\tau} e^{-\nu(t_0-s)}\left[\frac{(\tau\epsilon^{p_s})^{\gamma+\frac{3}{2}} + T_0^{\gamma+\frac{3}{2}}}{(s-t_0)_<^{\gamma/2}} + \frac{\epsilon^{p_s}}{(s-t_0)_<^{(\gamma+\rho)/2}} + \right.\right. \\ &\left.\left. \frac{T_1^{\gamma-\frac{1}{2}}T_0 e^{-\nu(s-t_0)} + T_0^{\gamma+\frac{1}{2}} e^{-(\gamma+\frac{1}{2})\nu(s-t_0)}}{(s-t_0)_<^{\gamma}}\right] ds\right).\end{aligned}$$

Since the singular terms in the denominator are all integrable for γ and ρ satisfying $\gamma < 1$ and $\gamma + \rho < 2$, we may neglect them in evaluating the integral, imposing the condition

$$\tau^{\gamma+\frac{3}{2}}\epsilon^{p_s(\gamma-\frac{1}{2})} \le 1, \tag{58}$$

we combine two terms in ϵ on the first line of the inequality above, and using the condition (55) to eliminate the term linear in T_0 on the second line, we find

$$T_0 \le c\left[T_0(0) + (e^{\nu\tau} - 1)\left(T_0^{\gamma+\frac{3}{2}} + \epsilon^{p_s}\right) + T_0^{\gamma+\frac{1}{2}}\right]. \tag{59}$$

The following Lemma turns this implicit inequality into a concrete bound on T_0.

Lemma 9. *Fix $\gamma \in (\frac{1}{2}, 1)$. For any $T_0 : \mathcal{C}\left((0,\tau), \mathbb{R}_+\right)$, if T_0 satisfies (59) then for any $\alpha_0 \in \left(0, \frac{1}{\gamma+3/2}\right)$ there exists $c > 0$, independent of $\epsilon > 0$, such that*

$$T_0(0) \le c\epsilon^{p_s\alpha_0(\gamma+\frac{1}{2})}, \tag{60}$$

implies

$$T_0 \le c\left(T_0(0) + \epsilon^{(1-\alpha_0)p_s}\right). \tag{61}$$

for all τ satisfying

$$\tau \le \frac{p_s\alpha_0}{\nu}\left|\ln\epsilon\right|. \tag{62}$$

Proof: We fix τ and define $a_0 = c\left(T_0(0) + e^{\nu\tau}\epsilon^{p_s}\right)$, $a_1 = c$, and $a_2 = ce^{\nu\tau}$. Introducing $f(x) = a_0 + a_1x^{\gamma+\frac{1}{2}} + a_2x^{\gamma+\frac{3}{2}}$, we recast (59) as $T_0 \le f(T_0)$. For $\alpha_0 < 1$ we have $a_0 \ll 1$, while $a_1 = \mathcal{O}(1)$. Assuming that $a_2 \ll a_0^{-1/(\gamma+\frac{1}{2})}$ on the whole interval $(0,\tau)$, the function f has a fixed point at $x_0(\tau) = a_0 + \mathcal{O}\left(a_1a_0^{\gamma+\frac{1}{2}} + a_2a_0^{\gamma+\frac{3}{2}}\right) = a_0 + o(a_0)$. Moreover $f'(x) < 1$ for $0 \le x \le x_1(\tau) = \left(a_2(\gamma+\frac{3}{2})\right)^{-(\gamma+\frac{1}{2})}$. From the bound on a_2 we see that $x_1 \gg 2a_0 \ge x_0$. Thus there is an excluded region; $x \le f(x)$ for $x > 0$ implies either $x \in (0, x_0(\tau))$ or $x > x_1(\tau)$, so long as a_2 is bounded as above. From the definition of a_2 and a_0 this last condition is met for τ satisfying (62) so long as $T_0(0)$ satisfies (60) and α_0 is constrained as in the Lemma. Since $T_0(0) < x_0$, it follows that $T_0(t) \le x_0(t)$ for all $t \in [0,\tau]$. The implication, $T_0(\tau) \le x_0$, is exactly (61). ∎

Remark 3. Of the conditions, (55), (58), and (62) imposed upon τ, the condition (62) is the most restrictive.

We have proven the following proposition.

Proposition 3. *Fix $\gamma \in (\frac{1}{2}, 1)$ and $\alpha_0 \in \left(0, \frac{1}{\gamma+3/2}\right)$. Then there exists $c > 0$ such that for any initial data W_n to (46-47) which satisfies*

$$\|W_n\|_{H_s^\gamma} \le c\epsilon^{p_s\alpha_0(\gamma+1/2)}, \tag{63}$$

then defining $t_{n+1} = t_n + \frac{p_s\alpha_0}{\nu}|\epsilon|$, we have the bound

$$\|W(t)\|_{H_s^\gamma} \le ce^{-\nu(t-t_n)}\left(\|W_n\|_{H_s^\gamma} + \epsilon^{(1-\alpha_0)p_s}\right), \tag{64}$$

valid for all $t \in I_n = [t_n, t_{n+1}]$.

Proof: Fix $\tau = \frac{p_s\alpha_0}{\nu}|\epsilon|$ in Lemma 9 and apply (61) to T_0 which satisfies the estimate (59). ∎

5.2. **Proof of Theorem 1.** We apply Propositions 2 and 3 to study the RG map $\mathcal{G}$ defined in (50). The map involves two stages, the evolution of $(W_n, \bar{z}_n)$ to $(W(t_{n+1}), z_0(t_{n+1}))$, and the projection of $(W(t_{n+1}), z_0(t_{n+1}))$ onto $(W_{n+1}, \bar{z}_{n+1})$. For the first stage, assuming W_n satisfies (63), we have

$$\|W(t_{n+1})\|_{H^\gamma_s} \le c\left(\epsilon^{p_s\alpha_0}\|W_n\|_{H^\gamma_s} + \epsilon^{p_s}\right).$$

For the second state, taking $\bar{z}_0 = \bar{z}_{n+1}$, $\tilde{z}_* = \bar{z}_n$, $z_* = z_0(t_{n+1})$, $W_* = W(t_{n+1})$, and $W_0 = W_{n+1}$ then (52) implies that

$$\begin{aligned}\left\|W_{n+1} - W(t^-_{n+1})\right\|_{H^\gamma_s} + \left|\bar{z}_{n+1} - z_0(t^-_{n+1})\right| &\le c\,|z_0(t_{n+1}) - \bar{z}_n|\ \|W(t^-_{n+1})\|_{H^\gamma_s},\\ &\le cT_1\left(\epsilon^{p_s\alpha_0}\|W_n\|_{H^\gamma_s} + \epsilon^{p_s}\right).\end{aligned}$$

However from (56) and (61) we see

$$\begin{aligned}T_1 &\le c(T_1^{\gamma+\frac12} + |\ln\epsilon|\epsilon^{p_s}) \le c\left(\|W_n\|^{\gamma+\frac12}_{H^\gamma_s} + \epsilon^{(1-\alpha_0)p_s(\gamma+\frac12)} + |\ln\epsilon|\epsilon^{p_s}\right),\\ &\le c\left(\|W_n\|^{\gamma+\frac12}_{H^\gamma_s} + |\ln\epsilon|\epsilon^{p_s}\right),\end{aligned}$$

where in the last inequality we imposed the condition $\alpha_0 < \frac{\gamma-\frac12}{\gamma+\frac12}$. Combining the two estimates on the jump we see that

$$\left\|W_{n+1} - W(t^-_{n+1})\right\|_{H^\gamma_s} + \left|\bar{z}_{n+1} - z_0(t^-_{n+1})\right| \le c\epsilon^{\alpha_0 p_s}\left(\|W_n\|_{H^\gamma_s} + \epsilon^{(1-\alpha_0)p_s}\right)^2,$$

so the jumps are quadratically small in terms of $\|W_n\|_{H^\gamma_s}$. Combining the estimates on the two stages of the RG map, we find

$$\|\mathcal{G}_1 W_n\|_{H^\gamma_s} \le c\left(\epsilon^{p_s\alpha_0}\|W_n\|_{H^\gamma_s} + \epsilon^{p_s}\right).$$

Defining $\eta_0 = \|W_0\|_{H^\gamma_s}$ then $\|W_n\|_{H^\gamma_s} \le \eta_n$ where $\{\eta_n\}_{n=0}^\infty$ are the iterates of the map $\eta_{n+1} = c(\epsilon^{\alpha_0 p_s}\eta_n + \epsilon^{p_s})$. It is easy to see that, for $\epsilon < \epsilon_0$ where ϵ_0 satisfies $c\epsilon_0^{\alpha_0 p_s} < 1$, that

$$\eta_n \to \eta_* = c\frac{\epsilon^{p_s}}{1 - c\epsilon^{\alpha_0 p_s}},$$

as $n \to \infty$. This is the condition which gives the bound on ϵ_0 in Theorem 1. Expressed in terms of W, the convergence of the η_n to η_* implies the bound

$$\|W(t)\|_{H^\gamma_s} \le c\left(e^{-\nu(t-t_0)}\|W_0\|_{H^\gamma_s} + \epsilon^{p_s}\right). \tag{65}$$

To recover the asymptotic pulse motion, we address the equality (54) for the pulse evolution and consider the situation where t is sufficiently large that $\|W\|_{H^\gamma_s} < c\epsilon^{p_s}$. In this regime we see from (56) that $T_1 \le c\epsilon^{p_s}|\ln\epsilon|$, and hence from (15) we have

$$\left|\left(R, \psi^\dagger_{\bar{z}_0}\right)_{L^2_s}\right| \le c(\sqrt{\epsilon}T_1 + \epsilon) \le c(\epsilon^{p_s+\frac12} + \epsilon).$$

Similarly from (53) we find

$$\left|\left(\Delta L\, W, \psi^\dagger_{\bar{z}_0}\right)\right|_{L^2_s} \le c\|\Delta L\, W\|_{H^\gamma_s}\|\psi^\dagger_{\bar{z}_0}\|_{H^\gamma_s} \le cT_1^{\gamma-\frac12}\|W\|_{H^\gamma_s} \le \epsilon^{2p_s}|\ln\epsilon|.$$

From the estimate (21) with $\beta = \gamma$ we see

$$\left|\left(\mathcal{N}(W), \psi^\dagger_{\bar{z}_0}\right)\right|_{L^2_s} \le c\|W\|^{\gamma+\frac12}_{H^\gamma_s} \le c\epsilon^{(\gamma+\frac12)p_s}.$$

Combining these estimates with equation (54) we obtain

$$z_0' = \epsilon^{p_s}\left(\eta_s, \psi^\dagger_{\bar{z}_0}\right)_{L^2_s} + \mathcal{O}\left(\epsilon^{(\gamma+\frac12)p_s}\right). \tag{66}$$

To complete the proof Theorem 1 we observe that the front evolution is smooth except for the jumps at the renormalization times. However these jumps are smaller than the error in the front velocity (66), and occur at long, $\mathcal{O}(|\ln \epsilon|)$, time intervals. We smooth out the jumps without affecting the curve velocity at leading order. Changing back to the y_0 variables in (66) using the transformations (8) and (9), yields (12). The result on the width of the ignition set follows from (22) and the estimate (11).

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