Preface

The purpose of this book is to describe the theory of Hankel operators, one of the most important classes of operators on spaces of analytic functions. Hankel operators can be defined as operators having infinite Hankel matrices (i.e., matrices with entries depending only on the sum of the coordinates) with respect to some orthonormal basis. Finite matrices with this property were introduced by Hankel, who found interesting algebraic properties of their determinants. One of the first results on infinite Hankel matrices was obtained by Kronecker, who characterized Hankel matrices of finite rank as those whose entries are Taylor coefficients of rational functions. Since then Hankel operators (or matrices) have found numerous applications in classical problems of analysis, such as moment problems, orthogonal polynomials, etc.

Hankel operators admit various useful realizations, such as operators on spaces of analytic functions, integral operators on function spaces on $(0, \infty)$, operators on sequence spaces. In 1957 Nehari described the bounded Hankel operators on the sequence space ℓ^2. This description turned out to be very important and started the contemporary period of the study of Hankel operators.

We begin the book with introductory Chapter 1, which defines Hankel operators and presents their basic properties. We consider different realizations of Hankel operators and important connections of Hankel operators with the spaces BMO and VMO, Sz.-Nagy–Foais functional model, reproducing kernels of the Hardy class H^2, moment problems, and Carleson imbedding operators.

It turns out that for the needs of applications it is also important to consider vectorial Hankel operators, i.e., Hankel operators on spaces of vector functions. We introduce vectorial Hankel operators in Chapter 2, to be used later in the book in control theory, approximation theory, and Wiener–Hopf factorizations (Chapters 11, 13, and 14).

In Chapter 3 we introduce another very important class of operators on spaces of analytic functions, the class of Toeplitz operators. They can be defined as operators having infinite matrices with entries depending only on the difference of the coordinates. Though Hankel and Toeplitz operators have quite different properties, Hankel operators play an important role in the study of Toeplitz operators, and vice versa. We also study in Chapter 3 vectorial Toeplitz operators.

In Chapter 4 we analyze the singular values of Hankel operators. The main result of the chapter is the Adamyan–Arov–Krein theorem, which shows that the nth singular value of a Hankel operator is the distance to the set of Hankel operators of rank at most n.

Chapter 5 deals with parametrization of solutions of the Nehari problem. In other words, we parametrize the symbols of a Hankel (or a vectorial Hankel) operator that belong to the ball in L^∞ of a given radius.

In Chapter 6 we describe the Hankel operators that belong to Schatten–von Neumann classes $\boldsymbol{S}_p$ as those whose symbols belong to certain Besov classes. We consider various applications of this description. In particular we obtain sharp results on rational approximation in the norm of BMO.

In this book we study many different applications of Hankel operators (in approximation theory, prediction theory, interpolation problems, control theory, etc). In Chapters 8 and 9 we use Hankel operators to study regularity conditions for stationary processes.

Chapter 10 is an introduction to the spectral theory of Hankel operators. We continue the analysis of the spectral problems of Hankel operators in Chapter 12, where we give a complete description of the spectral properties of the self-adjoint Hankel operators. It turns out that not only are Hankel operators used in control theory, but also the theory of Hankel operators can benefit from methods of control theory. In particular, in Chapter 12 the results on spectral properties of self-adjoint Hankel operators are based on balanced linear systems with continuous time and discrete time, a notion borrowed from control theory.

Chapter 11 is devoted to applications of Hankel operators in control theory. We consider linear systems with discrete time and continuous time, the problems of robust stabilization, model reduction, and model matching.

In Chapter 13 we study hereditary properties of maximizing vectors of vectorial Hankel operators. In other words, for a broad class of function spaces X we prove that if the symbol belongs to X, then all maximizing vectors belong to the same space X. We give several applications of this result. In particular, we use it to obtain hereditary properties of Wiener–Hopf factorizations.

In Chapter 14 vectorial Hankel operators are used in the theory of approximation by analytic matrix and operator functions. We introduce the important notion of superoptimal approximation and prove the uniqueness of a superoptimal approximation under certain mild conditions on the matrix function. We obtain certain special factorizations and prove inequalities between the singular values of the corresponding Hankel operators and superoptimal singular values of their symbols. This beautiful theory has been developed during the last decade; it demonstrates the importance of vectorial Hankel operators in noncommutative analysis.

One of the most beautiful applications of Hankel operators is given in Chapter 15. The last chapter gives a solution to the famous problem of whether a polynomially bounded operator on Hilbert space must be similar to a contraction. This problem remained open for a long time. In particular, it was one of the problems in a famous paper by Paul Halmos called "Ten problems in Hilbert space". Recently it has been solved in the negative with the help of vectorial Hankel operators.

In this book we discuss only classical Hankel operators (i.e., operators with Hankel matrices or, in other words, Hankel operators on the Hardy class H^2). For the last 20 years many interesting results have been obtained about various generalizations of Hankel operators (commutators of multiplications and Calderón–Zygmund operators, paracommutators, Hankel operators on Bergman spaces, Hankel operators on function spaces on the polydisk, on the unit ball in $\mathbb{C}^n$, on classical domains, etc). However, it is physically impossible to cover such generalizations in one book, and we restrict ourselves to classical Hankel operators.

Even under this constraint it is hardly possible to cover all aspects of Hankel operators and their applications (for example, this book does not include applications of Hankel operators in noncommutative geometry, perturbation theory, or asymptotics of Toeplitz determinants). Each chapter ends with Concluding Remarks, where the reader can find references to some results not included here.

Theorems, lemmas, and corollaries (as well as displayed formulas) are numbered lexicographically. Within the same chapter Theorem 3.5 means the fifth item of Section 3. To refer to a result from a different chapter, we use three numbers: Lemma 5.5.4 means the fourth item of Section 5 in Chapter 5. Reference to §6 means Section 6 within the same chapter. Reference to §4.3 means Section 3 in Chapter 4. Displayed formulas have an independent numeration.

For convenience we add two appendices in which the reader can find necessary information on operator theory and function spaces. Reference to Appendix 2.5 means Section 5 of Appendix 2.

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