Math 930 Problem Set 9

Due Wednesday, November 19

Problem (9.1) Use the inverse metric g^{ij} to form the trace of the Second Bianchi Identity to show:

(a) The contracted Bianchi Identity $div\, Ric = \frac{1}{2}\nabla s$, that is,

$$\nabla^j R_{ij} = \tfrac{1}{2}\nabla_i s \qquad \text{where} \;\; \nabla^m = g^{mn}\nabla_n.$$

(b) Show that the Einstein Tensor $G_{ij} = R_{ij} - \frac{1}{2} s g_{ij}$ is divergence-free:

$$\operatorname{div} G_{ij} = g^{ij}\nabla_j G_{ik} = 0.$$

Hint: Read pages 124-125 of the textbook.

Problem (9.2) Prove the following generalization of Myers' Theorem: Let M^n be a complete Riemannian manifold. Suppose that there are constants $a > 0$ and $b \geq 0$, and a function $f(s)$ such that, for all geodesic segments $\gamma(s)$ that minimize distance between their endpoints, we have

$$Ric(\dot{\gamma}(s)) \;\geq\; a + \frac{df}{ds}$$

along γ with $|f(s)| \leq b$. Then M is compact. (Note that this hypothesis does not imply that Ric is positive.)

Problem (9.3) Let G be a Lie group whose Lie algebra $\mathfrak{g}$ has trivial center (the *center* of $\mathfrak{g}$ is $\mathfrak{z} = \{X \in \mathfrak{g} \,|\, ad_X = 0\}$). In class we showed that G admits a bi-invariant Riemannian metric, and gave a formula for the curvature. Use Myers' Theorem to prove this converse:

If G (of the above type) admits a bi-invariant Riemannian metric, then G is compact and $\pi_1(G)$ is finite.