Math 930   Problem Set 8

Due Wednesday, November 19

Problem (8.1) Let (M, g) and (N, g') be Riemannian manifolds with Levi-Civita connections ∇ and ∇'. Consider the product $M \times N$ with the product metric $g \oplus g'$. Show that the Levi-Civita connection of $M \times N$ is $\nabla \oplus 1 + 1 \oplus \nabla'$, i.e. is given by

$$\nabla_{X+X'} Y + Y' \;=\; \nabla_X Y + \nabla'_{X'} Y'.$$

for all $X, Y \in \mathcal{T}(M)$ and $X', Y' \in \mathcal{T}(N)$.

Note that general vector fields on $M \times N$ are linear combinations of vector fields of the form $X + X'$ with coefficients in $C^\infty(M \times N)$.

Problem (8.2) The purpose of this problem is to show that the curvature does not determine the metric, even locally. Let M and N be the submanifolds of $\mathbb{R}^3$ parameterized by

$$f(s, \theta) = (s \sin\theta, s \cos\theta, \log s)$$

and

$$g(s, \theta) = (s \sin\theta, s \cos\theta, \theta),$$

both with the induced metrics.

(a) Show that the metric tensors of M and N are

$$\begin{pmatrix} 1 + s^{-2} & 0 \\ 0 & s^2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 + s^2 \end{pmatrix}$$

(b) Show that the curvature of both surfaces is $K(s, \theta) = \dfrac{-1}{(1 + s^2)^2}$.

Thus the map $\phi : M \to N$ by $\phi(f(s, \theta)) = g(s, \theta)$ preserves the curvature, but is not an isometry.

Hint: for both (a) and (b), follow the calculations done for the "Corrected Example 2" in Wednesday's class, but finding $K = \det S / \det g$ instead of H.

A *Cartan-Hadamard manifold* is a complete, simply-connected Riemannian manifold $(M.g)$ with non-positive sectional curvature $K_M \leq 0$. In particular, for each constant $C < 0$ and each dimension $n \geq 2$, the hyperbolic space $\mathbf{H}^n_R$ with $R = 1/\sqrt{-C}$ is such a space with $K = C$ (cf. Lee, pages 38 and 148).

Do the following problem, which is the very last problem in Lee's book:

Problem (8.3) Let M be a Cartan-Hadamard manifold with $K_M < C$ for some constant $C < 0$. Use Jacobi fields to prove that the volume of any geodesic ball is at least as large as the volume of the geodesic ball of the same radius in hyperbolic space of curvature C.