Math 930 Problem Set 6

Due Monday, October 27

Problem (6.1) Prove the second Bianchi identity

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0$$

for all vector fields X, Y, Z.

Problem (6.2) As in Problem 5.1, let $\{e_i\}$ be a local orthonormal frame of TM with dual frame $\{e^i\}$ of T^*M and a connection 1-form $\omega^i{}_j$ defined by $\nabla_X e^i = \sum \omega^i{}_j(X)\, e^j$. Show that in this frame, the curvature is the $\mathfrak{so}$(n)-valued 2-form

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

with $\Omega^i{}_j = \sum -R^i{}_{jkl}\, e^k \wedge e^l$ (note minus!). Use the conventions $\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$ and $R^i{}_{jkl} = \langle R(e_k, e_l)e^i, e^j \rangle$. (Note: here we are using the connection and curvature on T^*M).

Problem (6.3) A *locally symmetric space* is a Riemannian manifold (M^n, g) whose curvature tensor R satisfies $\nabla R = 0$.

(a) Prove that if M has constant sectional curvature then M is a locally symmetric space.

(b) Let $\gamma : [0,b] \to M$ be a geodesic in a locally symmetric space, and let X, Y, Z be parallel vector fields along γ. Prove that $R(X,Y)Z$ is parallel along γ.

(c) Prove that a connected, 2-manifold is locally symmetric if and only if it has constant sectional curvature.

Problem (6.4) Let $\gamma : [0,\infty) \to M$ be a geodesic in a locally symmetric space with tangent $T = \dot{\gamma}$. Let $v = T(0)$ be the tangent at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(w) = -R(v,w)v \qquad \text{for } w \in T_pM.$$

(a) Show that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_i\}$ of T_pM that diagonalizes K_v, so $K_v(e_i) = \lambda_i e_i$ for $i = 1, \ldots, n$. Extend these to vector fields $e_i(t)$ along γ by parallel transport. Prove that

$$K_T(e_i(t)) \;=\; \lambda_i\, e_i(t)$$

for all i and all $t \geq 0$, where λ_i is independent of t.

(c) Show that $X(t) = \sum_i x_i(t) e_i(t)$ is a Jacobi field along γ if and only if

$$\frac{d^2 x_i}{dt^2} \;+\; \lambda_i x_i \;=\; 0 \qquad i = 1, \ldots, n$$

The *conjugate points* of p along γ are those points $q = \gamma(t)$ for which there exists a Jacobi field along γ that vanishes at p and at q.

(d) Show that the conjugate points of p along γ are $\gamma(\pi k/\sqrt{\lambda_i})$, where k is an integer and λ_i is a positive eigenvalue of K_v.