Math 930 Problem Set 5

Due Wednesday, October 8

Problem 5.1 An *orthonormal moving frame* on an open set U in a Riemannian manifold is a set $\{e_1, \ldots, e_n\}$ of $n = \dim M$ vector fields on U that is everywhere orthonormal, i.e. $g(e_i, e_j) = \delta_{ij} \quad \forall i, j$. In calculations, it is often more convenient to use moving frames instead of local coordinates.

(a) Prove that each $p \in M$ has a neighborhood U such that there is an ON moving frame $\{e_i\}$ on U with $(\nabla_{e_i} e_j)_p = 0$ for all i, j. (Hint: Normal coordinates and Gram-Schmidt).

(b) Let $\{e_i\}$ be an ON moving frame. Show that the dual frame $\{e^i\}$ is an orthonormal set of 1-forms when the metic on T^*M is defined by $\langle \alpha, \beta \rangle = \sum_i \alpha(e_i)\beta(e_i)$.

(c) Given a moving frame, we define the *connection 1-forms* ω_i^j

$$\nabla_{e_i} e^j \;=\; \sum_k {\omega^j}_k(e_i)\; e^k.$$

Show that the connection is compatible with the metric if and only if ${\omega^j}_i = -{\omega^i}_j$, i.e. ${\omega^i}_j$ is a 1-form with values in the Lie algebra $\mathfrak{so}(n)$ of $n \times n$ skew-symmetric matrices.

Problem 5.2 Let N be a closed embedded submanifold of a complete Riemannnian manifold M. For any point $p \in M \setminus N$, we define the distance from p to N to be

$$d(p, N) \;=\; \inf\big\{ d(p, x) \,\big|\, x \in N \big\}.$$

(a) Use the continuity of $d(x, y)$ to prove that there is a point $q \in N$ that realizes this distance.

(b) The Hopf-Rinow Theorem then implies that there is a minimizing geodesic γ from p to q. Use the first variational formula (set up the variation carefully) to prove that γ intersects N orthogonally.

Problem 5.3 Do Problem 6.6 on page 113 of the textbook.

Problem 5.4 Let $E \to M$ be a vector bundle with connection ∇. The *covariant section derivative* of a section $\xi \in \Gamma(E)$ in the direction of vector fields X and Y is defined by

$$\nabla^2_{X,Y}\xi \;=\; \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi.$$

Prove that ∇^2 is tensorial in X and Y, i.e. $\nabla^2_{fX,Y}\xi = f\nabla^2_{X,Y}\xi$ and $\nabla^2_{X,fY}\xi = f\nabla^2_{X,Y}\xi$ for all $f \in C^\infty(M)$.