Math 930   Problem Set 4

Due Monday, September 29

Problem 4.1 Many interesting manifolds arise as *Riemannian submersions.* To understand this idea, do Problem 3.8 on page 45 of the textbook.

Problem 4.2 A smooth function $\phi : M \to \mathbb{R}$ determines a 1-form $d\phi$. If there is an affine connection on M then the covariant Hessian of ϕ is defined by

$$(\nabla^2\phi)(X, Y) \;=\; \nabla_X(\delta\phi)(Y).$$

(which differs slightly from the equation at the bottom of page 54 of the textbook).

(a) Show that $(\nabla^2\phi)(X, Y) \;=\; X \cdot Y\phi - (\nabla_X Y)\phi$.

(b) Prove that $\nabla^2\phi$ is a $\binom{2}{0}$ tensor.

(c) Show that a connection ∇ is torsion-free if and only if the covariant Hessian $\nabla^2\phi$ is a *symmetric* tensor for every $\phi \in C^\infty(M)$.

Problem 4.3 Do Problem 4.4 on page 63 of the textbook.

Problem 4.4 Do Exercise 5.7 on page 82 of the textbook.

Problem 4.5 Lobatchevski's non-euclidean geometry is the metric

$$g \;=\; \frac{1}{y^2}\Big(dx \otimes dx + dy \otimes dy\Big)$$

on the upper half-plane $H = \big\{(x, y) \in \mathbb{R}^2 \,|\, y > 0\big\}$ (thus $g_{11} = g_{22} = \frac{1}{y^2}$ and $g_{12} = 0$).

(a) Show that the Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = 0, \qquad \Gamma^2_{11} = \tfrac{1}{y}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\tfrac{1}{y}.$$

(b) Fix the vector $X_0 = (0, 1) \in T_{(0,1}H$ and let X_t be its parallel transport along the line $\gamma(t) = (t, 1)$. Show that X_t is the unit vector that makes an angle t with the positive x-axis, measured *clockwise.*

Hint: Since X_t is a unit vector and $y = 1$, we can write $X_t = (\cos\theta(t), \sin\theta(t))$ for some function $\theta(t)$. Write down the equation of parallel transport (using the above Γ^k_{ij}), and show that $\frac{d\theta}{dt} = -1$.