**Math 930 Problem Set 3**

Due Friday, September 19

**Problem 3.1** The energy of a path $\gamma : [a,b] \to M$ is

$$E(\gamma) = \int_a^b \|\dot\gamma\|^2 \, ds$$

where $\|X\|^2$ denotes $g(X,X)$. Use Hölder's inequality to prove that

$$L(\gamma)^2 \le (b-a)E(\gamma)$$

with equality if and only if $\gamma$ is parameterized proportionally to arclength.

**Problem 3.2** Let $E \to M$ be a vector bundle over a manifold. Use a partition of unity to prove that there exists a connection $\nabla$ on $E$.

*Suggestion: take $\nabla_{\partial/\partial x^i}$ to be $\frac{\partial}{\partial x^i}$ in each local trivialization.*

**Problem 3.3** Again, let $E \to M$ be a vector bundle, now with a metric $h$. Show that if a connection $\nabla$ on $E$ is compatible with the metric (i.e. $\nabla h = 0$)then, for each path $\gamma : [a,b] \to M$ from $p = \gamma(a)$ to $\gamma(b)$, the parallel transport operator $P_{\gamma,t} : E_p \to E_q$ is an isometry.

*Hint: Fix vectors $\xi, \eta \in E_p$, let $\xi(t) = P_{\gamma,t}\xi$ and $\eta(t) = P_{\gamma,t}\eta$ be their extensions to sections parallel along $\gamma$ and differentiate $\langle \xi_t, \eta_t \rangle = h(\xi_t, \eta_t)$ with respect to $t$.*

**Problem 3.4** Let $E$ and $\nabla$ be as in Problem 3.3. Show that parallel transport determines the connection in the following sense.

Given a vector field $X$ defined in a neighborhood of $p \in M$, let $\gamma(t)$ be a path in $M$ with $\gamma(0) = p$ and $\dot\gamma(0) = X(p)$. Show that for each section $\xi$ of $E$,

$$(\nabla_X \xi)_p \;=\; \lim_{t\to 0} \frac{P^{-1}_{\gamma,t}\xi(\gamma(t)) - \xi(p)}{t}.$$

**Start of proof.** Choose a local basis $\{e_\alpha\}$ of the fiber $E_p$ of $E$ at $p$, and extend it to a basis $\{e_\alpha(t)\}$ of $E_{(\gamma(t)}$ by parallel transport (so $\nabla_{\dot\gamma} e_\alpha = 0$), and then, for good measure, extend it to a neighborhood of $\gamma$. In this basis,

$$\xi(t) \;=\; \sum_\alpha \xi^\alpha(t)\, e_\alpha(t)$$

(a) Show that the parallel extension $\hat\xi(t) = P_{\gamma,t}\xi_0$ is given by keeping the coefficients $\xi^\alpha$ constant.

(b) Evaluate at $t = 0$ to get a expression for $(\nabla_X \xi)_p$ in this basis.

(c) Show that $P^{-1}_{\gamma,s}(\xi(t)) \;=\; \sum \xi^\alpha(t)\, e_\alpha(t-s)$.

(d) Now calculate the difference quotient.