Math 930 Problem Set 2

Due Friday, September 12

Problem 2.1. Let $\{e_1, \ldots, e_n\}$ be a basis of a vector space V and $\{f_i, \ldots, f_m\}$ a basis of W. We can then identify vectors in V with column vectors by

$$v = \sum v^i e_i \quad \longleftrightarrow \quad \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix}. \tag{0.1}$$

Given a linear map $L : V \to W$, we can write $Le_i = \sum_j A_i^j f_j$ for some numbers A_i^j, and identify

$$Lv \;=\; \sum_i v^i L(e_i) \;=\; \sum_{i,j} A_i^j v^i f_j \quad \longleftrightarrow \quad \begin{pmatrix} Av \end{pmatrix} \tag{0.2}$$

where A is the matrix with A_i^j in the j^{th} row and i^{th} column.

Now let $\{e^i\}$ and $\{f^j\}$ be the dual bases of V^* and W^*.

(a) For $\alpha \in V^*$, write down a correspondence similar to (0.1) above.

(b) Find a correspondence similar to (0.2) for the dual map $L^*\alpha$, using a matrix B.

(c) How are the matrices A and B related?

Problem 2.2. Work out the following special case of the Lemma on page 21 of Lee's book. A map

$$A : \mathcal{T}(M) \to C^\infty(M)$$

is called *tensorial* if it is linear over $C^\infty(M)$. Prove that such a map defines a smooth 1-form on M, as follows.

(a) Choose local coordinates $\{x^i\}$ and set $\alpha = \sum A_i(x)\, dx^i$ where A_i is obtained by applying A to the basis vector fields.

(b) Show that α is independent of coordinates by choosing coordinates $\{y^j\}$ and showing that the procedure in (a) defines the same 1-form α.

Problem 2.3 Let (M, g) be a Riemannian manifold with metric $g = f^*g_0$ induced by an embedding $f : M \to \mathbb{R}^m$ (Here $g_0(X, Y) = X \cdot Y$ is the euclidean metric on $\mathbb{R}^n$). Let $\{x^1, \ldots, x^n\}$ be coordinates on $U \subset M$, and write $f|_U$ as

$$\left(f^1(x^1, \ldots, x^n), f^2(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n)\right).$$

What is the matrix g_{ij} in terms of f?

Problem 2.4 Let H be the upper half-space in $\mathbb{R}^2$, i.e. $H = \{(x, y) | y > 0\}$. Let $f : H \to \mathbb{R}^2$ be the map $f(x, y) = (x, \log y)$. Find f^*g_0.