Math 930   Problem Set 1

Due Wednesday, September 2

Problem 1.1. (a) For a smooth function f, show that the Gaussian curvature of the level surface $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \,|\, z = f(x,y)\}$ is

$$\kappa(x,y) \;=\; \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

where f_x denotes $\frac{\partial f}{\partial x}$, f_{xx} denotes $\frac{\partial^2 f}{\partial x^2}$, etc. Follow these steps:

(i) Σ is the level set of $F(x,y) = f(x,y) - z$; calculate $\nu = \dfrac{\nabla F}{|\nabla F|}$ and the tangent vectors $u = \partial_x F$ and $v = \partial_y F$ to Σ.

(ii) Find the metric $g_{ij}(x,y)$ by calculating $g_{11} = u \cdot u$, $g_{12} = g_{21} = u \cdot v$, and $g_{22} = v \cdot v$.

(iii) Find the second fundamental form h_{ij} by

$$h_{11} = -\frac{\partial \nu}{\partial x} \cdot u \qquad h_{11} = -\frac{\partial \nu}{\partial x} \cdot v = -\frac{\partial \nu}{\partial y} \cdot u \qquad h_{11} = -\frac{\partial \nu}{\partial y} \cdot v.$$

(iv) The formula $h(X,Y) = g(SX,Y)$ shows that $h = gS$ as 2×2 matrices, so the Gaussian curvature is

$$K = \det S = \frac{\det h}{\det g}.$$

(b) Use the above formula to show that the Gaussian curvature of the saddle surface $\Sigma = \{z = x^2 - y^2\}$ is everywhere negative, and decays to 0 asymptotically like $\frac{1}{r^4}$ as $r = \sqrt{x^2 + y^2} \to \infty$.

Problem 1.2. This exercise gives practice with the index notation used in classical differential geometry. Treat it as formal algebra.

In a local coordinate system $\{x^i\}$ the Riemannian metric can be written as $g = g_{ij}\, dx^i\, dx^j$, where it is understood that we are summing over repeated indices ("Einstein summation notation"), that $g_{ij} = g_{ji}$, and that $dx^i\, dx^j$ means $dx^i \otimes dx^j$. In this setting, the *Christoffel symbols* are

$$\Gamma_{ij}^k \;=\; \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

(again using the summation convention, and where $g^{k\ell}$ denotes the entries to the inverse of the matrix whose entries are $g_{k\ell}$.

(a) Show that $\Gamma_{ij}^k = \Gamma_{ji}^k$, i.e. the Christoffel symbols are symmetric in their lower indices.

(b) Write the expression A_{ij}^k for Γ_{ij}^k at a point where $g_{ij} = \delta_{ij}$ (similar to the above, but a little simpler).

(c) Show that the A^k_{ij} from (b) are a6 solution to the equation

$$A^i_{kj} + A^j_{ki} - \frac{\partial g_{ij}}{\partial x^k} = 0.$$