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"Comparison Theorems in Riemannian Geometry"

CHAPTER 3

HOMOGENEOUS SPACES

In this chapter we shall study invariant metrics on homogeneous spaces – spaces on which a Lie group acts transitively. Homogeneous spaces are, in a sense, the nicest examples of riemannian manifolds and are good spaces on which to test conjectures.

We shall need some elementary facts about Lie groups, which we shall summarize without proof. The reader who is not familiar with this material should consult Chevalley [1946], Helgason [1962], Sternberg [1964]. We shall also use Frobenius' theorem and various properties of the Lie derivative. (See Sternberg [1964]).

3.1. Definition. A *Lie group* G is a smooth manifold (which we do not assume connected), which has the structure of a group in such a way that the map $\varphi : G\times G \to G$ defined by $\varphi(x, y) = x\cdot y^{-1}$ is smooth.

It can be shown that a C^∞ Lie group has a compatible real analytic structure (see Chevalley [1946]). Canonically associated to a Lie group is its Lie algebra.

3.2. Definition. A *Lie algebra* is a vector space V together with a map $[\ ,\ ]$: $V\times V\to V$ such that

(1) $[a_1 V_1 + a_2 V_2, W] = a_1[V_1, W] + a_2[V_2, W]$

(2) $[V, W] = -[W, V]$

(3) $[V_1, [V_2, V_3]] + [V_3, [V_1, V_2]] + [V_2, [V_3, V_1]] = 0.$

The last relation is called the Jacobi identity.

3.3. Example. If M is a smooth manifold, then $\chi(M)$ is a Lie algebra (of infinite dimension) with respect to the bracket operation $[X, Y](f) =$

$(XY-YX)f$. To check the Jacobi indentity is straightforward:

$$\begin{aligned}[X,[Y,Z]]&+[Z,[X,Y]]+[Y,[Z,X]]\\ &=X(YZ-ZY)-(YZ-ZY)X+Z(XY-YX)-(XY-YX)Z\\ &\quad+Y(ZX-XZ)-(ZX-XZ)Y\\ &=XYZ-XZY-YZX+ZYX+ZXY-ZYX-XYZ+YXZ\\ &\quad+YZX-YXZ-ZXY+XZY=0.\end{aligned}$$

We shall now describe the Lie algebra associated with a Lie group G. If G is a Lie group, we have for each $g\in G$ the diffeomorphisms $L_g : g_1\to gg_1$ and $R_g : g_1\to g_1 g$. We say that $V\in\chi(G)$ is *left invariant* (respectively right invariant) if $\mathrm{d}L_g(V(g_1))=V(gg_1)$ (respectively $\mathrm{d}R_g(V(g_1))=(V(gg_1))$. If V is left invariant, then it is uniquely determined by $V(e)$, where e is the identity element of G.

Conversely $V\in G_e=\mathfrak{g}$ gives rise to a left invariant vector field (l.i.v.f.) $V(g)=\mathrm{d}L_g(V(e))$. Since multiplication in G is smooth, so is a l.i.v.f. Therefore the l.i.v.f.'s form an n-dimensional subspace of $\chi(G)$, and we claim that the bracket of two l.i.v.f.'s is again left invariant. In fact it follows from the definition of Lie bracket that for any diffeomorphism $\varphi : M\to M$ and $X, Y\in\chi(M)$,

$$\mathrm{d}\varphi[X,Y]=[\mathrm{d}\varphi(X),\mathrm{d}\varphi(Y)].$$

Then

$$\mathrm{d}L_g[X,Y]=[\mathrm{d}L_g(X),\mathrm{d}L_g(Y)]=[X,Y]$$

if X, Y are l.i.v.f.'s. It follows that the l.i.v.f.'s form a Lie algebra $\mathfrak{g}$, the *Lie algebra of* G. Of course the choice of l.i.v.f. rather than r.i.v.f. is only a convention. The r.i.v.f.'s also form a Lie algebra isomorphic to the l.i.v.f.'s. It is often convenient to identify $\mathfrak{g}$ with G_e as above, and we will use both interpretations simultaneously. We note that as a consequence of this discussion, it follows that the tangent bundle of G is trivial. If $\overline{X}$ denotes the r.i.v.f. such that $\overline{X}(e)=X(e)$ for some l.i.v.f. X, then we shall see in Proposition 3.7 that

$$[\overline{X},\overline{Y}]|_e=[-\overline{X},-\overline{Y}]|_e=\overline{[X,Y]}|_e.$$

Hence the map $X\to-\overline{X}$ induces the isomorphism between the two Lie algebras.

3.4. Proposition. *Let $G_1\to G_2$ be a continuous homomorphism of Lie groups.*

Then φ is a real analytic map and hence induces $\mathrm{d}\varphi$, *which is a homomorphism of Lie algebras.*

It can be shown that if G_1 is simply connected and $f: G_{e_1}\to G_{e_2}$ is a homomorphism of Lie algebras, then there exists a unique analytic homomorphism $\varphi : G_1\to G_2$ such that $\mathrm{d}\varphi=f$. Also, any finite-dimensional Lie algebra is the Lie algebra of a simply connected Lie group. In this way the classification of simply connected Lie groups can be reduced to the algebraic problem of classification of Lie algebras. In case $\mathfrak{g}$ is semi-simple (defined before Proposition 3.39), this classification can be carried out explicitly. Finally, if G is a Lie group, any covering space of G is a Lie group in a natural way with Lie algebra isomorphic to $\mathfrak{g}$.

A subspace $\mathfrak{h}$ of $\mathfrak{g}$ which is closed under [,] is called a *subalgebra* of $\mathfrak{g}$. A subalgebra $\mathfrak{I}$ such that $[x,\mathfrak{I}]\subset\mathfrak{I}$ for all $x\in\mathfrak{g}$ is called an *ideal.*

3.5. Proposition. *If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ (considered as a Lie algebra of vector fields), then $\mathfrak{h}$ defines an involutive distribution and the maximal connected integral manifold H through e is a subgroup (which will not, in general, be a closed subset of G).* Conversely, if $H\subset G$ is a Lie subgroup (a subgroup which is also a* 1–1 *immersed submanifold), then the tangent space $\mathfrak{h}$ to H at e is a subalgebra of $\mathfrak{g}$. H is a normal subgroup if and only if $\mathfrak{h}$ is an ideal.*

As a special case of Proposition 3.5, we may take $\mathfrak{h}$ to be any 1-dimensional subspace of $\mathfrak{g}$. Then $[\mathfrak{h},\mathfrak{h}]=0\subset\mathfrak{h}$. The subgroup corresponding to such an $\mathfrak{h}$ is called a 1-*parameter subgroup*. For any $v\in G_e=\mathfrak{g}$, we have a natural homomorphism of Lie algebras $\mathrm{d}\varphi : R\to\mathfrak{g}$ with $\mathrm{d}\varphi(1)=v$, and hence a Lie group homomorphism $\varphi : R\to G$ mapping R onto the integral curve through the origin of the l.i.v.f. corresponding to V. We denote $\varphi(1)$ by e^v. Then $\mathrm{e}^{t_1 v}\,\mathrm{e}^{t_2 v}=\mathrm{e}^{(t_1+t_2)v}$. Moreover, there exists a neighborhood U of $0\in G_e$ such that $\mathrm{e} : U\to G$ is a diffeomorphism onto a neighborhood of the identity on G.

3.6. Proposition. *If $\varphi : G_1\to G_2$ is a homomorphism, then $\varphi(\mathrm{e}^v)=\mathrm{e}^{\mathrm{d}\varphi(v)}$.*

The following proposition will also be useful.

* H is in general only a 1–1 immersed submanifold, so its manifold topology is not always the relative topology.

3.7. Proposition.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}(e^{tx}e^{sy}e^{-tx})\Big|_{t=0,s=0} = [X,Y]|_e = -[\bar{X},\bar{Y}]|_e .$$

A word about the precise meaning of the expression in Proposition 3.7. For each fixed t, $e^{tx}e^{sy}e^{-tx}$ is a curve through the origin.

$$\frac{\partial}{\partial s}(e^{tx}e^{sy}e^{-tx})\Big|_{s=0}$$

is then a tangent vector in G_e. As we let t vary,

$$\frac{\partial}{\partial s}(e^{tx}e^{sy}e^{-tx})\Big|_{s=0}$$

describes a curve in G_e. It makes sense to differentiate this curve at $t=0$ and the result is a tangent vector in G_e.

Proof. Let φ_t be the 1-parameter group of diffeomorphisms generated by X. Then by the alternative definition of Lie bracket (see Sternberg [1964],

$$[X,Y] = \frac{d}{dt}\, d\varphi_{-t}(y(\varphi_t(e)))\Big|_{t=0} .$$

Now the integral curve Y of through $\varphi(t)$ is equal to $e^{tx}e^{sy}$ by the left invariance of Y. Then

$$[X,Y] = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi_{-t}(e^{tx}e^{sy})\Big|_{t=0,s=0} .$$

But for arbitrary $g\in G$, $\varphi_{-t}(g)$ is by definition the endpoint of the integral curve of X through g parameterized on the interval $[0,-t]$. By left invariance of the integral curves of X we then have

$$\varphi_{-t}g = g\,e^{-tx} = R_{e^{-tx}}(g) .$$

Hence

$$[X,Y] = \frac{\partial}{\partial t}\frac{\partial}{\partial s}(e^{tx}e^{sy}e^{-tx})\Big|_{t=0,s=0} .$$

The other equation follows similarly. □

3.8. Example. Let $M^n = \mathbf{R}^{n^2}$ denote the space of $n\times n$ matrices

$$GL(n) = \{m\in M \mid \det m \neq 0\} .$$

Since $\det(m_1\times m_2) = \det(m_1)\det(m_2)$ and $\det(I)=1$, it follows that $GL(n)$ is a group. $GL(n)$ is in a natural way an open subset of $\mathbf{R}^{n^2}$ and hence a manifold. Define

$$e^{tm} = I + tm + t^2\frac{m^2}{2!} + \cdots$$

The series can easily be shown to converge for all t to a continuous function satisfying

$$e^{t_1 v}e^{t_2 v} = e^{(t_1+t_2)v} .$$

Moreover, for any m,

$$e^{tr(m)} = \det(e^m)$$

(as follows easily by using the Jordan canonical form). It follows that the $\{e^{tm}\}$ are the parameter subgroups of $GL(n)$, and the Lie algebra $gl(n)$ of $GL(n)$ is naturally M^n. Now by Proposition 3.7,

$$\begin{aligned}[m_1,m_2] &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}(e^{tm_1}e^{sm_2}e^{-tm_1})\Big|_{t=0,s=0}\\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\{(I+tm_1+\cdots)(I+sm_2+\cdots)(I-tm_1+\cdots)\}\Big|_{t=0,s=0}\\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}(I+ts(m_1m_2-m_2m_1)+\cdots)\Big|_{t=0,s=0}\\ &= m_1m_2-m_2m_1 .\end{aligned}$$

We refer the reader to Chevalley [1946] for the description of the standard matrix subgroups of $GL(n)$.

3.9. Example. Myers and Steenrod [1939] have shown that the isometry group of any riemannian manifold is a Lie group. We will be using this fact implicitly below.

Given a Lie group G of dimension n, there is a natural homomorphism, the *adjoint representation*, from G to $GL(g)\simeq GL(n)$, defined as follows.

3.10. Definition. $\mathrm{Ad}_g(x) = \mathrm{d}R_g \circ \mathrm{d}L_{g^{-1}}(x)$.

Clearly $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1}\mathrm{Ad}_{g_2}$. Since for each g, the map $h \to ghg^{-1}$ is an automorphism of G, it follows that $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$,

$$\mathrm{Ad}_g([x, y]) = [\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)].$$

We set $\mathrm{ad} = \mathrm{d}(\mathrm{Ad})$. In other words $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the differential of the adjoint representation. We see from Proposition 3.7 that

$$\text{(3.11)} \qquad \mathrm{ad}_x(y) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}(\mathrm{e}^{tx}\,\mathrm{e}^{sy}\,\mathrm{e}^{-tx})\Big|_{t=0,\, s=0} = [x, y].$$

From the Jacobi identity it follows that ad_x is a derivation,

$$\mathrm{ad}_x[y, z] = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

From the remark after proposition 3.4 it follows that $\mathrm{Ad}\,\mathrm{e}^x = \mathrm{e}^{\mathrm{ad}_x}$.

We will now begin the study of homogeneous spaces.

3.12. Definition. If G is a connected Lie group and H a closed subgroup, G/H is the space of cosets $\{gH\}$, $\pi : G \to G/H$ is defined by $g \to [gH]$. G/H is called a *homogeneous space.*

Notice that for $h \in G$ we have $h(H) = [H]$ if and only if $h \in H$.

3.13. Proposition. *A subgroup which is a closed subset is an analytic submanifold and hence a Lie subgroup.*

3.14. Proposition. *G/H has a unique real analytic structure for which $\pi : G \to G/H$ is an analytic fibration.*

There is a natural smooth left action of G on G/H defined by $g_1[gH] = [g_1 gH]$. The diffeomorphism, Lg, of G/H induced by g will sometimes be denoted g. Since $g_1 g^{-1}[gH] = [g_1 H]$, the action of G is transitive; hence the terminology, homogeneous space. We want to study metrics on G/H for which G acts by isometries. Such metrics are called *invariant.* In G itself we may also consider the right action of G. Metrics invarient under both left and right actions are called *bi-invarient.* We should point out that invariant metrics do not exist for all G/H. Moreover, when they do exist, G may not be the full group $\hat{G}$ of isometries. $\hat{H}$, the largest group of isometries fixing some point $[gH] \in G/H$, is called the

isotropy group of that point. $\hat{H}$ is identified with a closed subgroup of th orthogonal group of $G/H_{[H]}$, and hence is compact. This identification come from the fact that on a connected riemannian manifold an isometry determined by its differential at a single point. It is easy to verify this b using the fact that isometries commute with the exponential map. That $\hat{H}$ is closed follows from the Cartan–Ambrose–Hicks Theorem.

G is said to act *effectively* on G/H if $L_g = 1$ (the identity map) implie $g = e$. Let H_0 be the largest subgroup of H which is normal in G. Set

$$G^* = G/H_0, \qquad H^* = H/H_0.$$

Then it is straightforward to check that G^*/H^* is diffeomorphic to G/H and that G^* acts effectively on G^*/H^*. This reduction may require som work in dealing with a specific example. If G acts effectively, then it may b identified with a Lie subgroup of $\hat{G}$. (Again this identification is 1–1 but no always an embedding.) In this case it is not hard to see that

$$\dim \hat{G} - \dim G = \dim \hat{H} - \dim H.$$

The tangent space to the point $[H]$ of G/H can be naturally identifie with $\mathfrak{g}/\mathfrak{h}$. Further, since the action of G on G and G/H commutes with π we have for $h \in H$ and $v \in \mathfrak{g}$

$$he^{tv}H = he^{tv}h^{-1}H.$$

Ad_H and $\mathrm{ad}_{\mathfrak{h}}$ leave $\mathfrak{h}$ invariant and hence act naturally on $\mathfrak{g}/\mathfrak{h}$. Differen tiating with respect to t yields

$$\text{(3.15)} \qquad \mathrm{d}L_h(v) = \pi(\mathrm{Ad}_h(v)),$$

where $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the natural projection.

3.16. Proposition. (1) *The set of G-invariant metrics on G/H is naturall isomorphic to the set of scalar products $\langle\ ,\ \rangle$ on $\mathfrak{g}/\mathfrak{h}$ which are invariant unde the action of* Ad_H *on* $\mathfrak{g}/\mathfrak{h}$.

(2) *If H is connected, a scalar product $\langle\ ,\ \rangle$ is invariant under* Ad_H *if an only if for each $h \in \mathfrak{h}$,* ad_h *is skew symmetric with respect to $\langle\ ,\ \rangle$.*

(3) *If G acts effectively on G/H, then G/H admits a G-invariant metric and only if the closure* $\mathrm{cl}\,(\mathrm{Ad}_H)$ *of the group* $\mathrm{Ad}_H \subset \mathrm{GL}(\mathfrak{g})$ *is compact.*

(4) *If G acts effectively on G/H, and if $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{p} \oplus$ with* $\mathrm{Ad}_H(\mathfrak{p}) \subset \mathfrak{p}$ *then G-invariant metrics on G/H are in 1–1 corresponden with* Ad_H*-invariant scalar products on $\mathfrak{p}$. These exist if and only if th closure of the group* $\mathrm{Ad}_H \mid \mathfrak{p}$ *is compact. Conversely, if G/H admits*

G-invariant metric, then G admits a left invariant metric which is right invariant under H, and the restriction of this metric to H is bi-invariant.

Setting $\mathfrak{p} = \mathfrak{h}^{\perp}$ *gives a decomposition as above.*

(5) *If H is connected, the condition* $\mathrm{Ad}_H(\mathfrak{p}) \subset \mathfrak{p}$ *is equivalent to* $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$.

(6) *If G is compact, then G admits a bi-invariant metric.*

Proof. (1) Given a left invariant metric on G/H, by restricting to the tangent space at $[H]$ we get an inner product on $\mathfrak{g}/\mathfrak{h}$. By (3.15), $\langle\ ,\ \rangle$ is invariant under Ad_H. Conversely, given such a $\langle\ ,\ \rangle$ we get an inner product on $G/H_{[H]}$. Given $[gH]$ we may define an inner product $\langle\ ,\ \rangle_{[H]}$ on $G/H_{[gH]}$ by setting

$$\langle x, y\rangle_{[gH]} = \langle \mathrm{d}L_{g^{-1}}(x), \mathrm{d}L_{g^{-1}}(y)\rangle_{[H]}.$$

Since

$$\begin{aligned}\langle \mathrm{d}L_{hg^{-1}}(x), \mathrm{d}L_{hg^{-1}}(y)\rangle_{[H]} &= \langle \mathrm{d}L_h \cdot \mathrm{d}L_{g^{-1}}(x), \mathrm{d}L_h \mathrm{d}L_{g^{-1}}(y)\rangle_{[H]} \\ &= \langle \mathrm{d}L_{g^{-1}}(x), \mathrm{d}L_{g^{-1}}(y)\rangle_{[H]}\end{aligned}$$

if $\langle\ ,\ \rangle$ is invariant under Ad_H, $\langle\ ,\ \rangle_{[gH]}$ is independent of which member of $[gH]$ we chose to define it. In this way we get a riemannian metric on G/H which is clearly left invariant.

(2) That the condition

$$\langle \mathrm{Ad}_{e^{tv}} x, \mathrm{Ad}_{e^{tv}} y\rangle = \langle x, y\rangle$$

for all $x,y \in \mathfrak{g}/\mathfrak{h}$ and $v \in \mathfrak{h}$ implies

$$\langle \mathrm{ad}_v x, y\rangle + \langle x, \mathrm{ad}_v y\rangle = 0$$

follows from Proposition 3.7. Conversely, if we assume the second condition, then by Proposition 3.6, for all x, y, v,

$$\begin{aligned}\langle \mathrm{Ad}_{e^{tv}} x, \mathrm{Ad}_{e^{tv}} y\rangle &= \langle e^{\mathrm{ad}_{tv}} x, e^{\mathrm{ad}_{tv}} y\rangle = \sum \langle e^{\mathrm{ad}_{tv}} x, (t^n/n!)(\mathrm{ad}_v)^n y\rangle \\ &= \sum \langle (-1)^n (t^n/n!)(\mathrm{ad}_v)^n e^{\mathrm{ad}_{tv}} x, y\rangle = \langle e^{\mathrm{ad}_{-tv}} e^{\mathrm{ad}_{tv}} x, y\rangle.\end{aligned}$$

Now the set of elements for which the claim holds obviously forms a closed subgroup $\bar{H} \subset H$. On the other hand, since every element of some open neighborhood U of the identity is of the form e^{tv}, the claim holds for elements of U. Thus the Lie algebra of $\bar{H}$ must be equal to that of H. Since H is connected, $\bar{H} = H$.

(3) Let G^*, H^* denote the isometry and isotropy groups of G/H. Since G acts effectively, we have a 1–1 homomorphism $G \to G^*$ inducing $\mathfrak{g} \to \mathfrak{g}^*$.

The group H^* is compact and therefore so is its image $\mathrm{Ad}_{H^*}: \mathfrak{g}^* \to \mathfrak{g}^*$. Let ω be a right invariant volume form on Ad_{H^*}, coming, for example, from a right invariant metric. Then for any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}^*$, define

$$\langle\!\langle x, y\rangle\!\rangle = \int_{\mathrm{Ad}_{H^*}} \langle \mathrm{Ad}_{h^*}(x), \mathrm{Ad}_{h^*}(y)\rangle\, \omega(h^*).$$

Then Ad_{H^*} acts by isometries with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ because

$$\langle\!\langle \mathrm{Ad}_{h_1}(x), \mathrm{Ad}_{h_1}(y)\rangle\!\rangle = \int_{\mathrm{Ad}_{H^*}} \langle \mathrm{Ad}_{h^*}\mathrm{Ad}_{h_1}(x), \mathrm{Ad}_{h^*}\mathrm{Ad}_{h_1}(y)\rangle\, \omega(h^*).$$

Since Ad is a homomorphism and ω is right invariant, this becomes

$$\begin{aligned}&\int_{\mathrm{Ad}_{H^*}} \langle \mathrm{Ad}_{h^* h_1}(x), \mathrm{Ad}_{h^* h_1}(y)\rangle\, \mathrm{d}R_{h_1^{-1}}\, \omega(h h_1) \\ &\qquad = \int_{\mathrm{Ad}_{H^*}} \langle \mathrm{Ad}_{h^*}(x), \mathrm{Ad}_{h^*}(y)\rangle\, \mathrm{d}R_{h_1^{-1}}\, \omega(h^*).\end{aligned}$$

Since $R_{h_1^{-1}}$ is a diffeomorphism, this becomes

$$\int_{\mathrm{Ad}_{H^*}} \langle \mathrm{Ad}_{h^*}(x), \mathrm{Ad}_{h^*}(y)\rangle\, \omega(h) = \langle\!\langle x, y\rangle\!\rangle.$$

Now the restriction of $\langle\!\langle\ ,\ \rangle\!\rangle$ to $\mathfrak{g}$ is an inner product with respect to which Ad_H acts by isometries. Hence Ad_H is contained in the (compact) orthogonal group with respect to this inner product, which implies that its closure is compact. Conversely, if the closure of Ad_H is compact, in a manner similar to the above we may construct an inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathfrak{g}$ such that Ad_H acts by isometries. Let $\mathfrak{p} = \mathfrak{h}^{\perp}$ with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. Then $\langle\!\langle\ ,\ \rangle\!\rangle|_{\mathfrak{p}}$ induces an Ad_H-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ under the identification $\rho: \mathfrak{p} \to \mathfrak{g}/\mathfrak{h}$.

(4) In view of (1), (3) this is straightforward to check.

(5) This follows as in (2).

(6) The proof is similar to that of (3). □

In Proposition 3.34 we will show that a simply connected Lie group H which admits a bi-invariant metric is the product of a compact group and a vector group $\mathbf{R}^k$ which is the center of H.

The following is an example of a homogeneous space which does not admit a left invariant metric.

3.17. Example.

$\mathrm{SL}(n)/\mathrm{SL}(n-1)$;

$\mathrm{SL}(n) = \{m \in M^n \mid \det m = 1\}$;

SL$(n-1)$ may be embedded in SL(n) by setting

$$m=\begin{bmatrix}1 & 0\\ 0 & m\end{bmatrix}$$

for all $m\in\mathrm{SL}(n-1)$;

SL(n) acts effectively on SL(n)/SL$(n-1)$.

The reader may verify that SL$(n-1)$ is not the product of a compact group and a vector group (see Proposition 3.34).

We are now going to compute the curvature of a left invariant metric on G/H. We begin with the special case of a left invariant metric on G itself. We use the notation A^* to denote the adjoint of the linear transformation A with respect to a given inner product.

3.18. Proposition. *Let $\langle\ ,\ \rangle$ be a left invariant metric on G, and let X,Y,Z be l.i.v.f.'s. Then:*

(1) $\nabla_X Y=\frac{1}{2}\{[X,Y]-(\mathrm{ad}_X)^*(Y)-(\mathrm{ad}_Y)^*(X)\}$;

(2) $\langle R(X,Y)Z,W\rangle=\langle\nabla_X Z,\nabla_Y W\rangle-\langle\nabla_Y Z,\nabla_X W\rangle-\langle\nabla_{[X,Y]}Z,W\rangle$;

(3) $$\langle R(X,Y)Y,X\rangle=\|[(\mathrm{ad}_X)^*(Y)+(\mathrm{ad}_Y)^*(X)]\|^2 -\langle(\mathrm{ad}_X)^*(X),(\mathrm{ad}_Y)^*(Y)\rangle-\tfrac{3}{4}\|[X,Y]\|^2 -\tfrac{1}{2}\langle[[X,Y],Y],X\rangle-\tfrac{1}{2}\langle[[Y,X],X],Y\rangle;$$

(4) 1-*parameter subgroups are geodesics if and only for all X,* $\mathrm{ad}_X^*(X)=0$.

Proof. By left invariance we have

$$0=X\langle Y,Z\rangle=\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle,$$
$$0=Y\langle X,Z\rangle=\langle\nabla_Y X,Z\rangle+\langle X,\nabla_Y Z\rangle,$$
$$0=Z\langle X,Y\rangle=\langle\nabla_Z X,Y\rangle+\langle X,\nabla_Z Y\rangle.$$

Computing as in Chapter 1, Section 0 gives

$$\langle\nabla_X Y,Z\rangle=\tfrac{1}{2}\{\langle[X,Y],Z\rangle-\langle Y,[X,Z]\rangle-\langle X,[Y,Z]\rangle\},$$

from which (1) readily follows.

(2) By left invariance, $X\langle\nabla_Y Z,W\rangle=0$. Therefore,

$$\langle\nabla_X\nabla_Y Z,W\rangle=-\langle\nabla_Y Z,\nabla_X W\rangle,$$
$$-\langle\nabla_Y\nabla_X Z,W\rangle=\langle\nabla_X Z,\nabla_Y W\rangle,$$
$$-\langle\nabla_{[X,Y]}Z,W\rangle=-\langle\nabla_{[X,Y]}Z,W\rangle.$$

(2) follows by adding these equations.

(3) follows from (1) and (2).

(4) is immediate from (1). □

In the case of a bi-invariant metric, (3) above simplifies considerably.

3.19. Corollary. *If $\langle\ ,\ \rangle$ is bi-invariant, then:*

(1) $\nabla_X Y=\frac{1}{2}[X,Y]$;

(2) $\langle R(X,Y)Z,W\rangle=\frac{1}{4}(\langle[X,W],[Y,Z]\rangle-\langle[X,Z],[Y,W]\rangle)$;

(3) $\langle R(X,Y)Y,X\rangle=\frac{1}{4}\|[X,Y]\|^2$.

In particular the sectional curvature is nonnegative.

(4) 1-*parameter subgroups are geodesics.*

Proof. Since $\langle\ ,\ \rangle$ is bi-invariant, by Proposition 3.16 we have

$$\langle Y,[X,Z]\rangle=-\langle Y,[Z,X]\rangle=\langle[Z,Y],X\rangle.$$

(1) now follows from the proof of (1) of Proposition 3.18.

(2) Substituting (1) into (2) of Proposition 3.18 gives

$$\langle R(X,Y)Z,W\rangle=\tfrac{1}{4}\langle[X,Z],[Y,W]\rangle-\tfrac{1}{4}\langle[Y,Z],[X,W]\rangle -\tfrac{1}{2}\langle[[X,Y],Z],W\rangle.$$

Using the Jacobi identity, the last term may be rewritten as

$$-\tfrac{1}{2}\langle[Y,[Z,X]],W\rangle-\tfrac{1}{2}\langle[X,[Y,Z]],W\rangle= =-\tfrac{1}{2}\langle[X,Z],[Y,W]\rangle+\tfrac{1}{2}\langle[Y,Z],[X,W]\rangle,$$

and (2) follows.

(3) follows immediately from (2).

(4) 1-parameter subgroups are the orbits of l.i.v.f.'s, so (1) implies (4). □

In order to generalize our formulas to the case of an arbitrary homogeneous space, we will prove a formula of O'Neill [1966], on the curvature of riemannian submersions. A *submersion* is a differentiable map $\pi: M^{n+k}\to N^n$ such that at each point $d\pi$ has rank n. It follows from the implicit-function theorem that $\pi^{-1}(p)$ is a smooth k-dimensional submanifold of M for all $p\in N$. Let V denote the tangent space to $\pi^{-1}(p)$ at $q\in\pi^{-1}(p)$. Assume that M and N have riemannian metrics, and set $H=V^{\perp}$. We call H and V the *horizontal* and *vertical* subspaces, respectively, and we use H and V as superscripts to denote horizontal and vertical components. π is called a *riemannian submersion* if $d\pi \mid H$ is an isometry. If X is a vector field on N, then there is a unique vector field $\bar{X}$ on M such that $\bar{X} \in H$ and $d\pi(\bar{X}) = X$. Also if $c: [0, 1] \to N$ is a piecewise smooth curve, and $q \in \pi^{-1}(c(0))$, then there is a unique curve $\bar{c}: [0, 1] \to M$ such that $\bar{c}(0) = q$, $\pi \cdot \bar{c} = c$, $\bar{c}'(t) \in H$. This follows from the theory of ordinary differential equations exactly as in the special case in which M is a principal bundle over N and H defines a connection; see Kobayashi and Nomizu [1963, 1969].

We now give a formula which relates the curvature $\bar{K}(\bar{X}, \bar{Y})$ of a plane section spanned by the orthonormal vectors $\bar{X}, \bar{Y}$ to that of the section spanned by X, Y at p. First of all, note that the expression $[\bar{X}, \bar{Y}]^V|_p$ depends only on the values of $\bar{X}$, $\bar{Y}$ at p. In fact, if T is a vector field tangent to V, then

$$\begin{aligned}\langle [\bar{X}, \bar{Y}], T\rangle &= \langle \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X}, T\rangle \\ &= \bar{X}\langle \bar{Y}, T\rangle - \langle \bar{Y}, \bar{\nabla}_{\bar{X}} T\rangle - \bar{Y}\langle \bar{X}, T\rangle + \langle \bar{X}, \bar{\nabla}_{\bar{Y}} T\rangle \\ &= \langle \bar{X}, \bar{\nabla}_{\bar{Y}} T\rangle - \langle \bar{Y}, \bar{\nabla}_{\bar{X}} T\rangle.\end{aligned}$$

3.20. Theorem (O'Neill). $K(X, Y) = \bar{K}(\bar{X}, \bar{Y}) + \frac{3}{4}\|[\bar{X}, \bar{Y}]^V\|^2$.

Thus riemannian submersions are curvature nondecreasing on the horizontal sections.

Let $\pi: M \to N$ be any smooth map. Vector fields $\tilde{X}, X$ are called *π-related* if at all points $q \in \pi^{-1}(p)$,

$$d\pi(\tilde{X}) = X.$$

In particular $\bar{X}, X$ as above are π-related. We shall make use of the following lemma.

3.21. Lemma. *If $\tilde{X}$,X are $\tilde{Y}$,Y are π-related, then $[\tilde{X}, \tilde{Y}]$ is π-related to $[X, Y]$.*

Proof. For any function $f: N \to \mathbf{R}$,

$$\tilde{X}(f \circ \pi) = d\pi(\tilde{X})(f) = X(f).$$

Therefore at $q \in \pi^{-1}(p)$,

$$d\pi([\tilde{X}, \tilde{Y}])f = ([\tilde{X}, \tilde{Y}])(f \circ \pi) = (\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X})(f \circ \pi) = (XY - YX)f = [X, Y](f).$$

Since a tangent vector is determined by its action on functions, the lemma follows. □

Proof of Theorem 3.20. Lemma 3.21, together with the riemannian submersion property, has the consequence that given X,Y,Z on N and a vertical field T on M,

$$\langle [\bar{X}, \bar{Y}], \bar{Z}\rangle = \langle [X, Y], Z\rangle, \qquad \langle [\bar{X}, T], \bar{Y}\rangle = 0. \tag{3.22}$$

Then the formula for the riemannian connection of Chapter 1, Section 0, together with (3.22) and the riemannian submersion property, gives

$$\begin{aligned}\langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z}\rangle &= \tfrac{1}{2}\{\bar{X}\langle \bar{Y}, \bar{Z}\rangle + \bar{Y}\langle \bar{X}, \bar{Z}\rangle - \bar{Z}\langle \bar{X}, \bar{Y}\rangle \\ &\quad + \langle [\bar{X}, \bar{Y}], \bar{Z}\rangle - \langle [\bar{X}, \bar{Z}], \bar{Y}\rangle - \langle [\bar{Y}, \bar{Z}], \bar{X}\rangle\} \\ &= \langle \nabla_X Y, Z\rangle,\end{aligned} \tag{3.23}$$

while if T is vertical,

$$\begin{aligned}\langle \bar{\nabla}_{\bar{X}} \bar{Y}, T\rangle &= \tfrac{1}{2}\{\bar{X}\langle \bar{Y}, T\rangle + \bar{Y}\langle \bar{X}, T\rangle - T\langle \bar{X}, \bar{Y}\rangle \\ &\quad + \langle [\bar{X}, \bar{Y}], T\rangle - \langle [\bar{X}, T], \bar{Y}\rangle - \langle [\bar{Y}, T], \bar{X}\rangle\}.\end{aligned}$$

Since $\langle \bar{X}, \bar{Y}\rangle = \langle X, Y\rangle$ and T is vertical, $T\langle \bar{X}, \bar{Y}\rangle = 0$. The first two terms on the right clearly vanish as do the last two, by (3.22). Therefore

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, T\rangle = \tfrac{1}{2}\langle [\bar{X}, \bar{Y}], T\rangle. \tag{3.24}$$

Thus by (3.23) and (3.24),

$$\bar{\nabla}_{\bar{X}} \bar{Y} = (\overline{\nabla_X Y}) + \tfrac{1}{2}[\bar{X}, \bar{Y}]^V. \tag{3.25}$$

Also, by (3.23) and (3.24),

$$\begin{aligned}\langle \bar{\nabla}_T \bar{X}, \bar{Y}\rangle &= \langle \bar{\nabla}_{\bar{X}} T, \bar{Y}\rangle + \langle [T, \bar{X}], \bar{Y}\rangle \\ &= -\langle \bar{\nabla}_{\bar{X}} \bar{Y}, T\rangle = -\tfrac{1}{2}\langle [\bar{X}, \bar{Y}], T\rangle = -\tfrac{1}{2}\langle [\bar{X}, \bar{Y}]^V, T\rangle.\end{aligned} \tag{3.26}$$

Now by (3.23) it is clear that

$$\bar{X}\langle \bar{\nabla}_{\bar{Y}} \bar{Z}, \bar{W}\rangle = X\langle \nabla_Y Z, W\rangle. \tag{3.27}$$

Therefore

$$\begin{aligned}\langle \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{Z}, \bar{W}\rangle &= \bar{X}\langle \bar{\nabla}_{\bar{Y}} \bar{Z}, \bar{W}\rangle - \langle \bar{\nabla}_{\bar{Y}} \bar{Z}, \bar{\nabla}_{\bar{X}} \bar{W}\rangle \\ &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \tfrac{1}{4}\langle [\bar{Y}, \bar{Z}], [\bar{X}, \bar{W}]\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \tfrac{1}{4}\langle [\bar{Y}, \bar{Z}]^V, [\bar{X}, \bar{W}]^V\rangle.\end{aligned} \tag{3.28}$$

Also by (3.23) and (3.26),

$$(3.29)\qquad \langle \bar\nabla_{[\bar X,\bar Y]}\bar Z,\bar W\rangle = \langle \bar\nabla_{[\bar X,\bar Y]^H}\bar Z,\bar W\rangle + \langle \bar\nabla_{[\bar X,\bar Y]^V}\bar Z,\bar W\rangle = \langle \nabla_{[X,Y]}Z,W\rangle - \tfrac12\langle [\bar Z,\bar W]^V,[\bar X,\bar Y]^V\rangle.$$

Therefore, using (3.28) and (3.29),

$$(3.30)\qquad \langle \bar R(\bar X,\bar Y)\bar Z,\bar W\rangle = \langle \bar\nabla_{\bar X}\bar\nabla_{\bar Y}\bar Z,\bar W\rangle - \langle \bar\nabla_{\bar Y}\bar\nabla_{\bar X}\bar Z,\bar W\rangle - \langle \bar\nabla_{[\bar X,\bar Y]}\bar Z,\bar W\rangle = \langle R(X,Y)Z,W\rangle + \tfrac14\langle [\bar X,\bar Z]^V,[\bar Y,\bar W]^V\rangle - \tfrac14\langle [\bar Y,\bar Z]^V,[\bar X,\bar W]^V\rangle + \tfrac12\langle [\bar Z,\bar W]^V,[\bar X,\bar Y]^V\rangle.$$

The theorem follows by setting $\bar Z=\bar Y$, $\bar X=\bar W$. □

We make one more general remark:

3.31. Proposition. *If $\pi: M\to N$ is a riemannian submersion, $\gamma:[0,1]\to N$ and $\bar\gamma:[0,1]\to M$ a horizontal lift, then γ is a geodesic if and only if $\bar\gamma$ is.*

Proof. From (3.25),

$$\bar\nabla_{\bar\gamma'}\bar\gamma' = \overline{\nabla_{\gamma'}\gamma'} + \tfrac12[\bar\gamma',\bar\gamma']^V = \overline{\nabla_{\gamma'}\gamma'},$$

and the claim follows immediately. □

Proposition 3.31 may be seen more geometrically from the relation

$$L[\varphi] = \int \|\varphi'\|\,\mathrm{d}t \geq \int \|(\varphi')^H\|\,\mathrm{d}t = L[\pi(\varphi)]$$

for any curve $\varphi:[0,1]\to M$, and the relation

$$L[\bar\gamma]=L[\gamma].$$

We now specialize back to the case of homogeneous spaces. The map $\pi: G\to G/H$ is a fibration and hence a submersion. If G/H admits a left invariant metric $\langle\!\langle\ ,\ \rangle\!\rangle$ then by Proposition 3.16(4), G admits a left invariant metric $\langle\ ,\ \rangle$ which is right invariant under H. The restriction of $\langle\ ,\ \rangle$ to $\mathfrak h$ is bi-invariant, and its restriction to $\mathfrak p=\mathfrak h^\perp$ induces $\langle\!\langle\ ,\ \rangle\!\rangle$. Then $\pi: G\to G/H$ is a riemannian submersion, and the curvature of G/H may be computed immediately from Proposition 3.18(3) and Theorem 3.20. The decomposition $\mathfrak g=\mathfrak p\oplus\mathfrak h$ corresponds precisely to the decomposition

$M_q=H\oplus V$ and for $X, Y\in\mathfrak p$, the correction term in Theorem 3.20 becomes $\frac34\|[X,Y]_{\mathfrak h}\|^2$. We get the formula

$$(3.32)\qquad K(X,Y) = \|(\mathrm{ad}_X)^*(Y)+(\mathrm{ad}_Y)^*(X)\|^2 - \langle(\mathrm{ad}_X)^*X,(\mathrm{ad}_Y)^*Y\rangle - \tfrac34\|[X,Y]_{\mathfrak p}\|^2 - \tfrac12\langle[[X,Y],Y],X\rangle - \tfrac12\langle[[Y,X],X],Y\rangle.$$

Here we have written $K(X,Y)$ for $K(\mathrm{d}\pi(X),\mathrm{d}\pi(Y))$. We will continue to do this.

Suppose that the metric $\langle\ ,\ \rangle$ on G is bi-invariant. In this case the corresponding metric on G/H is called *normal.* As in Corollary 3.19, the formula for the curvature simplifies substantially.

3.33. Corollary. *If the metric on G/H is normal, then*

(1) $K(X,Y)=\frac14\|[X,Y]_{\mathfrak p}\|^2+\|[X,Y]_{\mathfrak h}\|^2$.

In particular the sectional curvature is nonnegative.

(2) *The geodesics in G/H are the images of* 1*-parameter subgroups of G.*

Proof. (1) is immediate from Corollary 3.19(3) and Theorem 3.20. (2) Follows from Corollary 3.19(4) and Proposition 3.31. □

Berger [1961], has classified those normal homogeneous spaces which have *strictly positive* curvature. With two exceptions of dimension 7 and 13, they are precisely the symmetric spaces of rank 1 which will be introduced later in this chapter. Recently, Wallach [1972a, b] has discovered 3 new examples of (nonnormal) homogeneous spaces of positive curvature of dimensions 6, 12, 24. He has shown that in even dimensions these are the only nonsymmetric homogeneous spaces of positive curvature. However, he has also constructed infinitely many simply connected nondiffeomorphic 7-dimensional homogeneous spaces with positive curvature.

3.34. Proposition. *A simply connected Lie group which admits a bi-invariant metric is the product of a compact group and a vector group.*

Proof. Let $\mathfrak z$ denote the center of $\mathfrak g$

$$\mathfrak z=\{x\in\mathfrak g \mid [x,y]=0 \text{ for all } y\in\mathfrak g\}.$$

It is clear that $\mathfrak z$ is an ideal. On the other hand, if G admits a bi-invariant metric, and I_1 is any ideal of $\mathfrak g$, then $I_2=I_1^\perp$ is also an ideal since

$$0=\langle I_1,I_2\rangle=\langle[x,I_1],I_2\rangle=\langle I_1,[x,I_2]\rangle.$$

Therefore, in this case $\mathfrak{g}$ splits as $\mathfrak{g}=\mathfrak{z}\oplus\mathfrak{h}$. Let $G=Z\times H$ be the corresponding splitting of G. Then G is easily seen to be the isometric product of Z and H. Z is simply connected, abelian and therefore a vector group. The formula of Corollary 3.19 for the curvature of a Lie group with bi-invariant metric implies that if the corresponding Lie algebra has no center, then the Ricci curvature is strictly positive. Hence, by Myers' Theorem 1.26, H is compact. □

Theorems 8.17 and 8.21 are much more general results of the type of Proposition 3.34. They are proved by quite different methods.

The following example is due to Berger. In addition to illustrating how our formulas work in practice, it provides a counterexample to a certain conjecture about closed geodesics. This point is explained in Chapter 5.

3.35. Example (Berger). Consider the 3-dimensional Lie algebra L spanned by z_1, z_2, z_3 with multiplication table

$$[z_1, z_2] = -2z_3, \qquad [z_1, z_3] = 2z_2, \qquad [z_2, z_3] = -2z_1.$$

It is straightforward to check that the inner product defined by making z_1, z_2, z_3 orthonormal is invariant under ad L. It is known that the simply connected Lie group with Lie algebra L is the ordinary 3-sphere S^3 which is a 2-fold covering space of SO(3). In fact, by using the formula for the curvature, one may easily show that the sectional curvature is constant and equal to 1. We wish to consider the homogeneous space G/H with $G=S^3+\mathbf{R}$, and H the 1-parameter subgroup generated by $\alpha z_1+\beta z_4$, where $z_1\in L$, $\alpha^2+\beta^2=1$ and z_4 is the l.i.v.f. tangent to $\mathbf{R}$. If $\beta\neq 0$, then G/H is easily seen to be diffeomorphic to S^3. In fact $\Pi \mid S^3 x\{0\}$ is a non-singular smooth map from S^3 to G/H.

Further, $\Pi \mid S^3\times 0$ is trivially seen to be 1–1. (Notice that G does not act effectively on G/H). G/H may be given a normal metric by taking $\|z_4\|=1$ and $\langle z_i, z_4\rangle=0$, $i=1,2,3$. We note for future reference that $\gamma=\Pi(\exp t(-\beta z_1+\alpha z_4))$ is a geodesic, as follows from Corollary 3.33. In fact, γ is periodic of length $2\pi\beta$, since

$$\mathrm{e}^{2\pi\beta(-\beta z_1+\alpha z_4)} = \mathrm{e}^{2\pi(-\beta^2 z_1+\beta\alpha z_4)} = \mathrm{e}^{2\pi((\alpha^2-1)z_1+\beta\alpha z_4)}$$
$$= \mathrm{e}^{-2\pi z_1}\,\mathrm{e}^{2\pi\alpha(\alpha z_1+\beta z_4)} = \mathrm{e}^{2\pi\alpha(\alpha z_1+\beta z_4)}.$$

($\mathrm{e}^{-2\pi z_1}=\mathrm{e}$ since e^{tz_1} is a periodic geodesic of length 2π in S^3.)

Now if $z=\beta z_1-\alpha z_4$, then

$$\begin{aligned}
A &= \mu_1 z+\mu_2 z_2+\mu_3 z_3,\\
B &= \nu_1 z+\nu_2 z_2+\nu_3 z_3,\\
[A, B] &= 2\lambda_1 z_1+2\beta\lambda_2 z_2+2\beta\lambda_3 z_3,\\
\lambda_1 &= \mu_2\nu_3-\mu_3\nu_2,\\
\lambda_2 &= \mu_1\nu_3-\mu_3\nu_1,\\
\lambda_3 &= \mu_1\nu_2-\mu_2\nu_1.
\end{aligned}$$

Then one gets easily

$$\begin{aligned}
\|[A, B]_{\mathfrak{h}}\|^2 &= 4\alpha^2\lambda_1^2,\\
\|[A, B]\|^2 &= 4\beta^2(\lambda_1^2+\lambda_2^2+\lambda_3^2),\\
\|A\wedge B\|^2 &= \lambda_1^2+\lambda_2^2+\lambda_3^2>0,\\
K(A, B) &= \frac{(1+3\alpha^2)\lambda_1^2+\beta^2(\lambda_2^2+\lambda_3^2)}{\lambda_1^2+\lambda_2^2+\lambda_3^2}.
\end{aligned}$$

One sees that the sectional curvature satisfies

$$H=\beta^2\le K_M\le 1+3\alpha^2=4-3\beta^2=K.$$

We note that if $\beta^2/(4-3\beta^2)<\frac{1}{9}$, then

$$2\pi\beta<2\pi(4-3\beta^2)^{-1/2}=2\pi K^{-1/2}$$

as an easy computation shows. In Chapter 5 we will show that for an even-dimensional manifold M with $K\ge K_M\ge H>0$, every closed geodesic has length $\ge 2\pi K^{-1/2}$. If M is odd-dimensional with $K\ge K_M\ge\frac{1}{4}K$, the same result obtains. Our example, however, shows that such a result does not hold in general. The best one could hope for is $K\ge K_M\ge\frac{1}{9}K$.

We will now briefly discuss a special class of homogeneous spaces—the symmetric spaces.

3.36. Definition. The riemannian manifold M is called *locally symmetric* if for each $m\in M$ there exists r such that reflection through the origin (in normal coordinates) is an isometry on $B_r(m)$. M is *(globally) symmetric* if the above reflection extends to a global isometry $I_m: M\to M$.

3.37. Proposition. (1) *M is locally symmetric if and only if $\nabla R=0$.*

(2) *M simply connected complete and locally symmetric implies M symmetric.*

(3) *M symmetric implies M homogeneous.* $M=G/H$, *where G is the isometry group of M and H is the isotropy group of some point* $m\in M$.

(4) *Let* $M=G/H$ *be symmetric with G the isometry group of M and the isotropy group of* $m\in M$. *Let I denote the symmetry about m. Then* $g\to IgI$ *defines an automorphism* σ *of G such that* $\sigma^2=1$. *The set F of fixed points of* σ *is a closed subgroup containing H. Its identity component* F_0 *coincides with that of H.*

(5) *Conversely, let G be a Lie group,* σ *an automorphism such that* $\sigma^2=1$, *and* $\langle\ ,\ \rangle$ *a left invariant metric on G/F, where F is the set of fixed points of* σ. *The relation* $\sigma(gf)=\sigma(g)\,\sigma(f)=\sigma(g)f$ *shows that* σ *induces a diffeomorphism of G/F. If this diffeomorphism preserves* $\langle\ ,\ \rangle$, *then G/F is a symmetric space.**

(6) *A simply connected Lie group G possesses an automorphism* σ *such that* $\sigma^2=1$ *if and only if* $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{h}$ *with*

$$[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h},\qquad [\mathfrak{h},\mathfrak{p}]\subset\mathfrak{p},\qquad [\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}.$$

In case G/F admits a σ*-invariant riemannian metric, G/F is therefore globally symmetric and its curvature is given by*

$$K(X,Y)=\tfrac{1}{2}\langle[[X,Y],Y],X\rangle-\tfrac{1}{2}\langle[Y,X],X],Y\rangle,$$

where $X,Y\in\mathfrak{p}$ *are orthonormal.*

Proof. (1) Since I_m is an isometry, $\mathrm{d}I_m$ commutes with ∇R. Hence

$$-[\nabla_X R](y,z)\,w=\mathrm{d}I_m([\nabla_X R](y,z)w)=[\nabla_{-X}R](-y,-z)(-w)$$
$$=[\nabla_X R](y,z)w\,.$$

The converse, which is an easy consequence of Lemma 1.35, is left as an exercise (or see Helgason [1962]).

(2) This follows immediately from the Cartan–Ambrose–Hicks Theorem, using the condition $\nabla R=0$ as above.

(3) Given a geodesic segment $\gamma:[0,t]\to M$,

$$\gamma(0,t)\cup I_{\gamma(t)}(\gamma(0,t))\cup\cdots$$

is γ extended arbitrarily far. Hence M is complete by the Hopf–Rinow

* It can be shown that even if G/F does not admit a σ-invariant metric it nonetheless admits a unique left invariant affine connection preserved by σ. Thus in general G/F is *affine symmetric*.

Theorem 1.8. Given p,q, let $\gamma:[0,t_0]\to M$ be a geodesic segment from p to q. Then $I_{\gamma(t_0/2)}(p)=q$. Hence M has a transitive group of isometries and is homogeneous.

(4) The only nontrivial part is to show $H_0=F_0$. Since $H\subset F$ it suffices to check that $F_0\subset H$. For $f\in F$, we have $f(m)=I\circ f\circ I(m)=I\circ f(m)$. However, m is the only point of a normal coordinate ball $B_r(m)$ which is fixed by I. Hence there is a neighborhood U of e in G such that $F\cap U\subset H$. It follows that $F\cap H$ is open (since $H\subset F$). But $F\cap H$ is also closed since F and H are closed. Hence $F_0=H_0$.

(5) The symmetry about a coset $[gF]$ is given by $L_g\circ\sigma L_{g-1}$.

(6) If $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{h}$ as above, then the linear map $\mathrm{d}\sigma$ defined by

$$\mathrm{d}\sigma\mid\mathfrak{h}=1,\qquad \mathrm{d}\sigma\mid\mathfrak{p}=-1$$

is easily seen to be an automorphism of $\mathfrak{g}$ such that $(\mathrm{d}\sigma)^2=1$. It induces an automorphism σ of G by the remarks following Proposition 3.4. Conversely given such a σ, take $\mathfrak{h}$, $\mathfrak{p}$ to be its $+1$ and -1 eigenspaces, respectively. Then for example

$$\mathrm{d}\sigma([p_1,p_2])=[\mathrm{d}\sigma(p_1),\mathrm{d}\sigma(p_2)]=[-p_1,-p_2]=[p_1,p_2]$$

shows $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}$, and the other relations follow similarly.

If $X,Y,Z\in\mathfrak{p}$, then $[X,Z]\in\mathfrak{h}$ and

$$\langle(\mathrm{ad}_X)^*(Y),Z\rangle=\langle Y,[X,Z]\rangle=0.$$

Therefore $(\mathrm{ad}_X)^*\,Y\subset\mathfrak{h}$. Now if $T\in\mathfrak{h}$,

$$\langle Y,[T,X]\rangle=-\langle[T,Y],X\rangle$$

because the metric on G is right invariant under H. Therefore

$$\langle(\mathrm{ad}_X)^*(Y),T\rangle=\langle Y,[X,T]\rangle=-\langle[Y,T],X\rangle=-\langle(\mathrm{ad}_Y)^*(X),T\rangle.$$

Therefore, substituting in (3.32) gives

$$K(X,Y)=-\tfrac{1}{2}\langle[[X,Y],Y],X\rangle-\tfrac{1}{2}\langle[[Y,X],X],Y\rangle.\ \square$$

A complete classification of symmetric spaces is available (Helgason [1962]). In particular, the only simply connected symmetric spaces having positive curvature are the spheres of constant curvatures, complex and quaternionic projective spaces, and the Cayley plane. These are sometimes referred to as *rank one* symmetric spaces, and except for the spheres they have canonical metrics with sectional curvature varying between $\frac{1}{4}$ and 1.

As an example, we will compute the curvature of complex projective space. The calculations for the other rank one spaces are similar.

3.38. Example. The unitary group $U(n)$ is defined as the (compact) group of $n\times n$ matrices with complex entries (α_{ij}) such that $(\alpha_{ij})^{-1}=(\overline{\alpha_{ji}})$. The special unitary group $\mathrm{SU}(n)$ is the subgroup of $U(n)$ of matrices of determinant 1. The Lie algebra $\mathfrak{u}(n+1)$ of $U(n+1)$ consists of skew hermitian matrices $(\alpha_{ij})=(-\overline{\alpha_{ji}})$. For $\mathfrak{su}(n+1)$ we must add the condition

$$\mathrm{trace}(\alpha_{ij})=0.$$

Complex projective space is the homogeneous space $\mathrm{CP}(n)=\mathrm{SU}(n+1)/U(n)$, where $U(n)$ is embedded in $\mathrm{SU}(n+1)$ as

$$\left[\begin{array}{c|c} U & \mathbf{0} \\ \hline \mathbf{0} & \overline{\det U} \end{array}\right],$$

where $U\in U(n)$. Geometrically, $\mathrm{CP}(n)$ may be thought of as the collection of 1-dimensional complex subspaces of $\mathbf{C}^{n+1}$.

The following may be easily checked. The rule $\langle A, B\rangle = -\frac{1}{2}\,\mathrm{trace}(AB)$ defines a bi-invariant metric on $\mathrm{SU}(n+1)$ which gives rise to the decomposition $\mathfrak{su}(n+1)=\mathfrak{p}+\mathfrak{u}(n)$, where $\mathfrak{p}$ consists of matrices of the form

$$\alpha=\left[\begin{array}{c|c} \mathbf{0} & \begin{matrix}\bar\alpha_1\\ \vdots \\ \bar\alpha_n\end{matrix} \\ \hline \alpha_1\cdots\alpha_n & \mathbf{0} \end{array}\right].$$

$\mathfrak{p}$ may be thought of as a complex n-space or real $2n$-space. Multiplication by i gives a real linear transformation $J:\mathfrak{p}\to\mathfrak{p}$ such that $J^2=-\mathrm{I}$ and $\langle x, y\rangle=\langle J(x), J(y)\rangle$. $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{u}(n)$, so that $\mathrm{CP}(n)$ is a symmetric space.

Now if $\|\alpha\|=\|\beta\|=1$ and $\langle\alpha,\beta\rangle=0$, by Corollary 3.33,

$$\langle R(\alpha,\beta)\beta,\alpha\rangle=\|[\alpha,\beta]\|^2,$$

where

$$[\alpha,\beta]=\left[\begin{array}{c|c} \beta_i\bar\alpha_j-\alpha_i\bar\beta_j & 0 \\ \hline 0 & \sum_i \alpha_i\bar\beta_i-\beta_i\bar\alpha_i \end{array}\right].$$

$$\begin{aligned}\|[\alpha,\beta]\|^2 &= -\tfrac{1}{8}\sum_{ij}(\beta_i\bar\alpha_j-\alpha_i\bar\beta_j)(\beta_j\bar\alpha_i-\alpha_j\bar\beta_i)\\ &\quad -\tfrac{1}{8}\sum_{ij}(\alpha_i\bar\beta_i-\beta_i\bar\alpha_i)(\alpha_j\bar\beta_j-\beta_j\bar\alpha_j)\\ &= \sum_i\alpha_i\bar\alpha_i\sum_j\beta_j\bar\beta_j-\tfrac{1}{2}\sum_i\beta_i\bar\alpha_i\sum_j\beta_j\bar\alpha_j\\ &\quad -\tfrac{1}{2}\sum_i\alpha_i\bar\beta_i\sum_j\alpha_j\bar\beta_j-\\ &\quad -\tfrac{1}{2}\sum_i(\beta_i\alpha_i-\beta_i\bar\alpha_i)\sum_j(\alpha_j\bar\beta_j-\beta_j\bar\alpha_j)\\ &= \tfrac{1}{4}(\|\alpha\|^2\,\|\beta\|^2)+\tfrac{3}{4}\langle J(\alpha),\beta\rangle^2=\tfrac{1}{4}+\tfrac{3}{4}\langle J(\alpha),\beta\rangle^2,\end{aligned}$$

where the last step follows from the relations

$$\begin{aligned}\sum(\alpha_i\bar\beta_i+\beta_i\bar\alpha_i) &= \langle\alpha,\beta\rangle,\\ \sum(-\alpha_i\bar\beta_i+\beta_i\bar\alpha_i) &= \mathrm{i}\langle J(\alpha),\beta\rangle.\end{aligned}$$

If $\langle J(\alpha),\beta\rangle=1$, then $K(\alpha,\beta)=1$, while if $\langle J(\alpha),\beta\rangle=0$, $K(\alpha,\beta)=\frac{1}{4}$.

Given a Lie algebra $\mathfrak{g}$, we define the *Killing form* as the form

$$B(g_1, g_2)=\mathrm{trace}(\mathrm{ad}\,g_2\;\mathrm{ad}\,g_1).$$

Then B is easily seen to be symmetric. Further, for all $x\in\mathfrak{g}$, $\mathrm{ad}\,x$ is skew symmetric with respect to B; i.e.

$$B(\mathrm{ad}_x\,y_1, y_2)=-B(y_1,\mathrm{ad}_x\,y_2).$$

If B is nondegenerate, we say that $\mathfrak{g}$ is *semi-simple.* It is easy to show that this implies that $\mathfrak{g}$ is a direct sum of *simple ideals.* An ideal is called *simple* if it contains no proper ideal. It is also true that if $\mathfrak{g}$ is a direct sum of simple ideals, then $\mathfrak{g}$ is semi-simple; but this is more difficult to prove; see Helgason [1962].

3.39. Proposition. (1) *If the group G is compact, the Killlng form B of $\mathfrak{g}$ is negative semi-definite. If B is negative definite, G is compact.*

(2) *If G is semi-simple and noncompact, then $\mathfrak{g}$ may be decomposed as $\mathfrak{p}\oplus\mathfrak{h}$ such that $B\mid\mathfrak{h}$ is negative definite, $B\mid\mathfrak{p}$ is positive definite, $\mathfrak{h}$ is a (maximal compact) subalgebra,* $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$, $[\mathfrak{h},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}$.

(3) *Let G be semi-simple and non-compact, and let $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{h}$ be as above. Then $B\mid\mathfrak{p}$ defines a metric invariant under* $\mathrm{d}\sigma$ *so that G/H becomes a symmetric space. The curvature of G/H is given by*

$$K(X, Y)=-\|[X, Y]\|^2.$$

(4) *Let G be compact semi-simple, $\sigma^2 = I$, and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ the decomposition into the + and − eigenspaces of σ. Then $-B \mid \mathfrak{g}$ is a bi-invariant metric and defines a symmetric metric on G/H. The curvature is given by $K(X, Y) = \|[X, Y]\|^2$.*

(5) *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be as in* (3) *and let $[\ ,\ ]$ denote the Lie algebra structure. Then the rule*

$$[\![p_1, p_2]\!] = -[p_1, p_2], \qquad [\![p, h]\!] = [p, h], \qquad [\![h_1, h_2]\!] = [h_1, h_2] \tag{3.40}$$

defines a new Lie algebra structure $[\![\ ,\ ]\!]$ on g. Let $\hat{B}$ denote the Killing form of $[\![\ ,\ ]\!]$. Then

$$\hat{B} \mid \mathfrak{p} = -B \mid \mathfrak{p}, \qquad \hat{B}(\mathfrak{p}, \mathfrak{h}) = 0, \qquad \hat{B} \mid \mathfrak{h} = B \mid \mathfrak{h}.$$

In particular, B is negative definite.

Conversely, given $[\![\ ,\ ]\!]$, define $[\ ,\ ]$ by (3.40). *Then $[\ ,\ ]$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ are as in* (2).

(6) *If G is a compact Lie group, then $G = (G \times G)/G$ is a symmetric space, where $G \subset G \times G$ is the diagonal inclusion.*

Proof. (1) If G is compact, then it admits a bi-invariant metric $\langle\ ,\ \rangle$. For all g, Ad_g is in the orthogonal group of $\langle\ ,\ \rangle$. Therefore, for all x, ad_x is represented by a skew symmetric matrix. But the trace of the square of a non-zero skew symmetric matrix is negative. If, on the other hand, B is negative definite, then $-B$ induces a bi-invariant metric on G. Using the fact that $\mathfrak{g}$ has no center, the claim follows from Proposition 3.33.

(2) See Helgason [1962], p. 156.

(3) Everything but the formula for the curvature follows immediately from (2). Moreover, if $X, Y \in \mathfrak{p}$, then $[[X, Y], Y] \in \mathfrak{p}$, and by Proposition 3.37(6),

$$\begin{aligned} K(X, Y) &= -\tfrac{1}{2}\langle[[X, Y], Y], X\rangle - \tfrac{1}{2}\langle[[Y, X], X], Y\rangle \\ &= -\tfrac{1}{2}B([[X, Y], Y], X) - \tfrac{1}{2}B([[Y, X], X], Y) \\ &= +\tfrac{1}{2}B([X, Y], [X, Y]) + \tfrac{1}{2}B([Y, X], [Y, X]). \end{aligned}$$

But $[X, Y] \in \mathfrak{h}$ and $B \mid \mathfrak{h} = -\langle\ ,\ \rangle \mid \mathfrak{h}$. Therefore the above becomes $-\|[X, Y]\|^2$.

(4) This follows by an argument completely analogous to that of (3).

(5) This is straightforward, and we omit the details.

(6) Set $\sigma(g_1, g_2) = (g_2, g_1)$. □

The metric in the Berger example is, of course, $-B$. The symmetry about e in G considered as a symmetric space is given by $g \to g^{-1}$.

A pair of symmetric spaces related as in (5) are called *dual* to one another.

We note that in a locally symmetric space that since $\nabla R = 0$, the Jacobi equation $\nabla_T \nabla_T J = R(T, J)T$ has constant coefficients and hence may be solved explicitly. In fact, if $T, E_2, \ldots, E_n$ are an orthonormal base of eigenvectors of $x \to R(T, x)T$, at $t = 0$, then solutions vanishing at $t = 0$ are of the form

$$\sin(t\sqrt{\lambda})\, E(t), \qquad \sinh(t\sqrt{-\lambda})\, E(t), \qquad tE(t)$$

according as $\lambda > 0$, $\lambda < 0$, $\lambda = 0$, where $E(t)$ is a parallel eigenvector with eigenvalue λ. In particular, if G is a compact Lie group then

$$R(T, J)T = -\tfrac{1}{4}(\mathrm{Ad}\, T)^2(J).$$

Since the nonzero eigenvalues of the square of a skew symmetric matrix occur in pairs, we have:

3.41. Proposition (Bott). *If G is a compact Lie group with bi-invariant metric, then all conjugate points are of even order.*

The following result will be utilized in Chapter 5 in the proof that in a symmetric space geodesics minimize up to the first conjugate point. Suppose that G is compact, $\sigma: G \to G$, $\sigma^2 = 1$ and H is the fixed point set of σ. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \qquad d\sigma(\mathfrak{h}) = \mathfrak{h}, \qquad d\sigma(\mathfrak{p}) = -\mathfrak{p}.$$

Call $g \in G$ a *transvection* if $g \in \exp \mathfrak{p} = \mathfrak{T}$. $g \in \mathfrak{T}$ clearly implies $\sigma(g) = g^{-1}$, but not conversely.

3.42. Proposition. *The map $\Phi: [gH] \to g\,\sigma(g^{-1})$ defines an imbedding of G/H onto $\mathfrak{T}$. If $\langle\ ,\ \rangle$ is a bi-invariant metric, then $\mathfrak{T}$ is a totally geodesic submanifold, and the metric on G/H induced by Φ is 2 times the normal metric.*

Proof. (1) Φ is well defined:

$$\begin{aligned} gh\,\sigma((gh)^{-1}) &= gh\,\sigma(h^{-1}g^{-1}) = gh\,\sigma(h^{-1})\,\sigma(g^{-1}) = ghh^{-1}\,\sigma(g^{-1}) \\ &= g\,\sigma(g^{-1}). \end{aligned}$$

(2) Φ is injective:

$$g\,\sigma(g^{-1}) = f\,\sigma(f^{-1})$$

implies

$$f^{-1}g=\sigma(f^{-1})\,(\sigma(g^{-1}))^{-1}=\sigma(f^{-1})\,\sigma(g)=\sigma(f^{-1}g).$$

This implies $f^{-1}g\in H$ or $[fH]=[gH]$.

(3) $\Phi(G/H)=\mathfrak{T}$: G/H is compact and therefore complete. Hence $\exp_{[H]}\mid G/H_{[H]}$ is onto. $\mathrm{d}\pi:\mathfrak{p}\to G/H_{[H]}$ is injective. By Corollary 3.33, 1-parameter subgroups of G are geodesics and project to geodesics in G/H. Therefore $\pi\circ\exp$ maps $\mathfrak{p}$ onto G/H. Given g let $\mathrm{e}^p\in\mathfrak{T}$ be such that $[gH]=[\mathrm{e}^pH]$. Then

$$\Phi(g)=\Phi(\mathrm{e}^p)=\mathrm{e}^p\,\sigma(\mathrm{e}^{-p})=\mathrm{e}^{2p}\in\mathfrak{T}$$

Thus $\Phi(G/H)\subset\mathfrak{T}$. But given $g=\mathrm{e}^p\in\mathfrak{T}$,

$$\Phi(\mathrm{e}^{p/2})=\mathrm{e}^{p/2}\,\Phi(\mathrm{e}^{-p/2})=\mathrm{e}^p=g.$$

(4) Let $B_r(\mathrm{e})$ be a normal coordinate ball. Then

$$B_r(\mathrm{e})\cap\mathfrak{T}=\exp_{\mathrm{e}}(B_r(0)\cap\mathfrak{p}).$$

Clearly $\exp_{\mathrm{e}}(B_r(0)\cap\mathfrak{p})\subset B_r(\mathrm{e})\cap\mathfrak{T}$.

Conversely if $g\in B_r(\mathrm{e})\cap\mathfrak{T}$, then σ maps the unique minimal geodesic

$$\mathrm{e}^{tx}=\exp tx:[0,1]\to M$$

from e to g into the unique minimal geodesic from e to $g^{-1}=\exp(-x)=\mathrm{e}^{-x}$. But this geodesic is just e^{-tx}, so that $g=\mathrm{e}^x$ with $\mathrm{d}\sigma(x)=-x$.

(5) $\Phi\circ\pi(g)=g^2$ if $g\in\mathfrak{T}$. Thus $\mathrm{d}\Phi\circ\mathrm{d}\pi\mid\mathfrak{p}=2I$, where I is the identity map. In fact,

$$\Phi\circ\pi(g)=g\,\sigma(g^{-1})=gg=g^2.$$

(6) Φ is an imbedding and the metric on G/H induced by Φ is 2 times the normal metric: At $[H]$ this follows from (5), so it suffices to prove the relation

$$\mathrm{e}^p\,\Phi([\mathrm{e}^{-p}g])\,\mathrm{e}^p=\Phi(g)$$

for all $\mathrm{e}^p\in\mathfrak{T}$. For then, taking $[\mathrm{e}^pH]=[gH]$ as is possible by (5), the above implies

$$\mathrm{d}R_g\circ\mathrm{d}L_g\circ\mathrm{d}\Phi_{\mathrm{e}}=\mathrm{d}\Phi_g.$$

But, in fact,

$$\mathrm{e}^p\,\Phi([\mathrm{e}^{-p}g])\,\mathrm{e}^p=\mathrm{e}^p\mathrm{e}^{-p}\,g\sigma(g^{-1}\mathrm{e}^p)\,\mathrm{e}^p=g\sigma(g^{-1})\,\mathrm{e}^{-p}\mathrm{e}^p=\Phi(g).$$

(7) $\mathfrak{T}$ is a totally geodesic submanifold: By (3), (4) it follows that if $B_r(\mathrm{e})$ is a normal coordinate ball then $\mathfrak{T}\cap B_r(\mathrm{e})$ is a submanifold which is

geodesic at e. Therefore it suffices to verify that $\mathrm{e}^{p/2}\mathfrak{T}\mathrm{e}^{p/2}=\mathfrak{T}$. But if $\mathrm{e}^x\in\mathfrak{T}$, then by (3),

$$\mathrm{e}^{p/2}\,\mathrm{e}^x\,\mathrm{e}^{p/2}=\Phi(\mathrm{e}^{p/2}\,\mathrm{e}^{x/2})\in\mathfrak{T}.$$

This completes the proof. □

We shall now briefly describe some applications of Theorem 3.20 to the problem of finding examples of manifolds which admit a metric of nonnegative curvature but are not diffeomorphic to a homogeneous space.

3.43. Example. Let G be a Lie group, K a compact subgroup and $\langle\ ,\ \rangle$ a metric right invariant under K. Let M be a manifold on which K acts by isometries. Then K acts by isometries on the product $G\times M$ by $k(g,m)=(gk^{-1},km)$. Clearly K acts without fixed point, and the quotient, which we write as $G\times_K M$, is a manifold.

$\Pi: G\times M\to G\times_K M$ is a submersion. Topologically, $G\times_K M$ is the bundle with fibre M, associated with the principal fibration $K\to G\to G/K$. Since K acts by isometries, $G\times_K M$ naturally inherits a metric such that Π becomes a riemannian submersion. If the metrics on G and M have nonnegative curvature, then by Theorem 3.20 so does the metric on $G\times_K M$. For G compact, we could take the bi-invariant metric on G, which does have nonnegative curvature. Even if M is homogeneous, if K does not act transitively then $G\times_K M$ is not, in general, homogeneous; see Cheeger [1973] for details.

3.44. Example (Gromoll and Meyer). Let $\mathrm{Sp}(n)$ denote the n-dimensional symplectic group realized by $n\times n$ matrices Q, with quaternionic entries, satisfying $Q\bar{Q}^t=I$. $\mathrm{Sp}(n)$ is a compact Lie group and carries a bi-invariant metric. Let $\mathrm{Sp}(1)$ act on $\mathrm{Sp}(2)$ by

$$Q\to\begin{bmatrix}q&0\\0&q\end{bmatrix}Q\begin{bmatrix}q&0\\0&1\end{bmatrix},\qquad q\in\mathrm{Sp}(1),\quad Q\in\mathrm{Sp}(2).$$

This action is free and the quotient space turns out to be an exotic 7-sphere. Thus, this exotic sphere carries a metric of nonnegative curvature. A calculation shows that on an open dense set of points the curvature is actually strictly positive.

At present all known examples of manifolds of nonnegative curvature are constructed by techniques closely related to the above.