Math 930 Problem Set 6

Problem (6.1) (a) Prove that if $\phi : G \to H$ is a Lie group homomorphism, then the differential $\phi_* : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

(b) As a special case, show that if $H \subset G$ is a Lie subgroup, then the Lie algebra $\mathfrak{h}$ of H is a vector subspace of $\mathfrak{g}$ that is closed under brackets.

Problem (6.2) Consider the basis

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

for the Lie algebra $\mathfrak{su}(2)$. For each positive real number a, define a left-invariant metric g on $SU(2)$ by declaring aX, Y, Z to be an orthonormal frame. Compute the sectional curvatures of this metric for the planes spanned by $(X, Y), (Y, Z)$ and (Z, X).

Remark: $SU(2)$ is diffeomorphic to S^3 by the map that sends $(\alpha, \beta) \in S^3 \subset \mathbb{C}^2$ to $\begin{pmatrix} \alpha & \beta \\ -\beta & \overline{\alpha} \end{pmatrix} \in SU(2)$. These metrics are called the *Berger metrics.*

Problem (6.3) Let G be a Lie group whose Lie algebra $\mathfrak{g}$ has trivial center (the *center* of $\mathfrak{g}$ is $\mathfrak{z} = \{X \in \mathfrak{g} \,|\, ad_X = 0\}$). In class we showed that G admits a bi-invariant Riemannian metric, and gave a formula for the curvature. Use Myers' Theorem to prove this converse:

If G (of the above type) admits a bi-invariant Riemannian metric, then G is compact and $\pi_1(G)$ is finite.

Problem (6.4) Let G be a Lie group with a bi-invariant metric g, and let $H \subset G$ be a Lie subgroup.

(a) Show that H is totally geodesic (see the definition below).

(b) If H is connected, show that it is flat in the induced metric if and only if it is Abelian.

Definition: A submanifold N of a Riemannian manifold (M, g) is called *totally geodesic* if for every $p \in N$ and $X \in T_pN$, the geodesic $\gamma(t)$ in M with initial condition $\gamma(0) = p$, $\dot{\gamma}(0) = X$, lies in N for all t.