Math 930 Problem Set 5

Due Wednesday, October 19

Problem 5.1 Let $\{e_i\}$ be a local orthonormal frame of TM with dual frame $\{e^i\}$ of T^*M. Define the *connection 1-forms* ω_i^j of the Levi-Civita connection by

$$\nabla_X e^i = \sum \omega^i{}_j(X)\, e^j.$$

(a) Show that the connection is compatible with the metric if and only if $\omega^j{}_i = -\omega^i{}_j$, i.e. $\omega^i{}_j$ is a 1-form with values in the Lie algebra $\mathfrak{so}(n)$ of $n \times n$ skew-symmetric matrices.

(b) Show that the curvature is the $\mathfrak{so}$(n)-valued 2-form

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

with $\Omega^i{}_j = -\sum R^i{}_{jkl}\, e^k \wedge e^l$ (note minus!). Use the conventions that $R^i{}_{jkl} = \langle R(e_k, e_l)e^i, e^j\rangle$ and $(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$.

Problem (5.2) A *locally symmetric space* is a Riemannian manifold (M^n, g) whose curvature tensor R satisfies $\nabla R = 0$.

(a) Prove that if M has constant sectional curvature then M is a locally symmetric space.

(b) Let $\gamma : [0, b] \to M$ be a geodesic in a locally symmetric space, and let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is parallel along γ.

(c) Prove that a connected 2-manifold is locally symmetric if and only if it has constant sectional curvature.

Problem (5.3) Let $\gamma : [0, \infty) \to M$ be a geodesic in a locally symmetric space with tangent $T = \dot{\gamma}$. Let $v = T(0)$ be the tangent at $p = \gamma(0)$. Define a linear transformation $K_v : T_pM \to T_pM$ by

$$K_v(w) = -R(v, w)v \qquad \text{for } w \in T_pM.$$

(a) Show that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_i\}$ of T_pM that diagonalizes K_v, so $K_v(e_i) = \lambda_i e_i$ for $i = 1, \dots, n$. Extend these to vector fields $e_i(t)$ along γ by parallel transport. Prove that

$$K_T(e_i(t)) \;=\; \lambda_i\, e_i(t)$$

for all i and all $t \geq 0$, where λ_i is independent of t.

(c) Show that $X(t) = \sum_i x_i(t)e_i(t)$ is a Jacobi field along γ if and only if

$$\frac{d^2 x_i}{dt^2} \;+\; \lambda_i x_i \;=\; 0 \qquad i = 1, \dots, n$$

The *conjugate points* of p along γ are those points $q = \gamma(t)$ for which there exists a Jacobi field along γ that vanishes at p and at q.

(d) Show that the conjugate points of p along γ are $\gamma(\pi k/\sqrt{\lambda_i})$, where k is an integer and λ_i is a positive eigenvalue of K_v.