Math 930 Problem Set 4

Problem 4.1 Following the derivation of the First Variational Formula for the energy function $E(\gamma)$, write down all steps in the derivation of the corresponding formula for the length function $L(\gamma)$.

The end result is given on page 207 of the textbook using very ugly notation.

Problem 4.2 Let N be a closed embedded submanifold of a complete Riemannnian manifold M. For any point $p \in M \setminus N$, we define the distance from p to N to be

$$d(p, N) \;=\; \inf \left\{ d(p, x) \,\middle|\, x \in N \right\}.$$

(a) Use the continuity of $d(x, y)$ to prove that there is a point $q \in N$ that realizes this distance.

The Hopf-Rinow Theorem (which we will discuss later) then implies that there is a length-minimizing geodesic γ from p to q. As we have seen, this geodesic is also energy-minimizing.

(b) Use the First Variational Formula (set up the variation carefully) to prove that γ intersects N orthogonally.

Problem 4.3 Let $E \to M$ be a vector bundle with connection ∇. The *covariant section derivative* of a section $\xi \in \Gamma(E)$ in the direction of vector fields X and Y is defined by

$$\nabla^2_{X,Y}\xi \;=\; \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi. \tag{1}$$

(a) Prove that ∇^2 is tensorial in X and Y, i.e. $\nabla^2_{fX,Y}\xi = f\nabla^2_{X,Y}\xi$ and $\nabla^2_{X,fY}\xi = f\nabla^2_{X,Y}\xi$ for all $f \in C^\infty(M)$.

(b) Derive this formula by:

  (i) Explain how to regard $\nabla\xi$ as a section of the bundle $T^*M \otimes E$, and write a formula for $\nabla\xi$ in a local moving frame $\{e_\alpha\}$.

  (ii) Write a similar formula for $\nabla^2\xi = \nabla(\nabla\xi)$ as a section of $T^*M \otimes T^*M \otimes E$, and show how your formula yields formula (1) above.