Math 930 Problem Set 3

Problem 3.1 Recall that the gradient of $f \in C^\infty(M)$ is the unique vector field ∇f such that

$$df(X) = g(\nabla f, X)$$

for all vector fields X on (M, g). Use this formula to show that ∇f is given in local coordinates $\{x^i\}$ by

$$\nabla f \;=\; \sum_{i,j} \left[g^{ij} \frac{\partial f}{\partial x^i} \right] \frac{\partial}{\partial x^j}.$$

Problem 3.2 In class we showed that the Laplacian of a function f is given in local coordinates by

$$\Delta f \;=\; -\frac{1}{\sqrt{\det g_{ij}}}\, \partial_i \left(g^{ij} \sqrt{\det g_{ij}}\, \partial_j f \right).$$

Use Lemma 10 from class to rewrite this as

$$\Delta f \;=\; -g^{ij}\partial_i\partial_j f + \Gamma^i \partial_i f \qquad \text{where } \Gamma^i = \pm g^{jk}\Gamma^i_{jk} \text{ (determine sign).}$$

Problem 3.3 Let $E \to M$ be a vector bundle over a manifold. Use a partition of unity to prove that there exists a connection ∇ on E.

Suggestion: take $\nabla_{\partial/\partial x^i}$ to be $\frac{\partial}{\partial x^i}$ in each local trivialization.

Problem 3.4 Again, let $E \to M$ be a vector bundle, now with a metric h. A connection ∇ on E is compatible with the metric if $\nabla h = 0$, or equivalently, if

$$X \cdot h(\xi, \eta) \;=\; h(\nabla_X \xi, \eta) \;+\; h(\xi, \nabla_X \eta) \qquad \forall X \in \mathrm{Vect}(M),\ \xi, \eta \in \Gamma(E).$$

Such connections always exist (by a slight modification of your proof in the previous problem — no need to show this).

Show that if ∇ is compatible with h then, for each path $\gamma : [a, b] \to M$ from $p = \gamma(a)$ to $\gamma(b)$, the parallel transport operator $P_{\gamma,t} : E_p \to E_q$ is an isometry.

Hint: Fix vectors $\xi, \eta \in E_p$, let $\xi(t) = P_{\gamma,t}\xi$ and $\eta(t) = P_{\gamma,t}\eta$ be their extensions to sections parallel along γ and differentiate $h(\xi_t, \eta_t)$ with respect to t.