Math 930 Problem Set 2

Problem 2.1. Work out the following special case of Lemma 2 stated in class. A map

$$A : \Gamma(TM) \to C^\infty(M)$$

is called *tensorial* if it is linear over $C^\infty(M)$. Prove that such a map defines a smooth 1-form on M, as follows.

(a) Choose local coordinates $\{x^i\}$ and set $\alpha = \sum A_i(x)\, dx^i$ where A_i is defined as $A(\frac{\partial}{\partial x^i})$.

(b) Show that α is independent of coordinates by choosing coordinates $\{y^j\}$ and showing that the procedure in (a) defines the same 1-form α.

Problem 2.2 Let (M, g) be a Riemannian manifold with metric $g = f^* g_0$ induced by an embedding $f : M \to \mathbb{R}^m$ (Here $g_0(X, Y) = X \cdot Y$ is the euclidean metric on $\mathbb{R}^n$). Let $\{x^1, \ldots, x^n\}$ be coordinates on $U \subset M$, and write $f|_U$ as

$$\big(f^1(x^1, \ldots, x^n), f^2(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n)\big)\,.$$

What is the matrix g_{ij} in terms of f?

Problem 2.3 Let H be the upper half-space in $\mathbb{R}^2$, i.e. $H = \big\{(x, y)|y > 0\big\}$. Let $f : H \to \mathbb{R}^2$ be the map $f(x, y) = (x, \log y)$. Find $f^* g_0$.

Problem 2.4 The energy of a path $\gamma : [a, b] \to M$ is

$$E(\gamma) = \int_a^b \|\dot\gamma\|^2 \; ds$$

where $\|X\|^2$ denotes $g(X, X)$. Use Hölder's inequality (including the statement about when equality holds) to prove that

$$L(\gamma)^2 \le (b - a)E(\gamma)$$

with equality if and only if γ is parameterized proportionally to arclength.