THIN COMPACTIFICATIONS AND VIRTUAL FUNDAMENTAL CLASSES

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Abstract. We define a notion of virtual fundamental class that applies to moduli spaces in gauge theory and in symplectic Gromov-Witten theory. For universal moduli spaces over a parameter space, the virtual fundamental class specifies an element of the Čech homology of the compactification of each fiber; it is defined if the compactification is "thin" in the sense that its boundary has homological codimension at least two.

The moduli spaces that occur in symplectic Gromov-Witten theory and in many gauge theories are often orbifolds that can be compactified by adding "boundary strata" of lower dimension. Often, it is straightforward to prove that each stratum is a manifold, but more difficult to prove "collar theorems" that describe the structure of neighborhoods of the boundary strata. The lack of collar theorems is an impediment to applying singular homology to the compactified moduli space, and in particular to defining its fundamental homology class. The purpose of this paper is to show that collar theorems are not needed to define a virtual fundamental class as an element of Čech homology. Indeed, existing results in the literature are enough to prove the existence of virtual fundamental classes in some cases.

There are two classes of homology theories, exemplified by singular homology and by Čech homology. We will use two Čech-type theories: Čech and Steenrod homologies. These have two features that make them especially well-suited for applications to compactified moduli spaces:

(1) For any closed subset A of a locally compact Hausdorff space X, the relative group $H_p(X, A)$ is identified with $H_p(X \smallsetminus A)$. As Massey notes [Ma2, p. vii]:

> ... one does not need to consider the relative homology or cohomology groups of a pair (X, A); the homology or cohomology groups of the complementary space $X - A$ serve that function. In many ways these "single space" theories are simpler than the usual theories involving relative homology groups of pairs. The analog of the excision property becomes a tautology, and never needs to be considered. It makes possible an intuitive and straightforward discussion of the homology and cohomology of a manifold in the top dimension, without any assumption of differentiability, triangulability, compactness, or even paracompactness!

(2) Čech homology satisfies a "continuity property" ((1.9) below) that allows one to define virtual fundamental classes by a limit process.

We briefly review Steenrod homology in Section 1. Then, in Section 2, we apply Property (1) to manifolds M that admit compactifications $\overline{M}$ for which the "boundary" $\overline{M} \smallsetminus M$ is "thin" in the sense that it has homological codimension at least 2. There may be many such

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compactifications. If M is oriented and d-dimensional, every thin compactification carries a fundamental class

$$[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M};\mathbb{Z})$$

in Steenrod homology. This class pushes forward under maps $M \to Y$ that extend continuously over $\overline{M}$, and many properties of the fundamental classes of manifolds continue to hold.

We then enlarge the setting by considering thinly compactified families. For this we start with a Fredholm map

$$\begin{array}{c} \mathcal{M} \\ \big\downarrow \pi \\ \mathcal{P} \end{array} \tag{0.1}$$

between Banach manifolds of index d that extends to a proper map $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ so that the boundary $\mathcal{S} = \overline{\mathcal{M}} \setminus \mathcal{M}$ is stratified by Banach manifolds of index at most $d-2$ (see Definition 3.2). The fiber $\overline{\mathcal{M}}_p$ over each regular value $p \in \mathcal{P}$ is then a thin compactification in the sense of Section 2, so has a fundamental class, *which we now regard as an element of Čech homology* (see Lemma 1.1). Because regular values are dense, a limiting process using Property (2) then yields a class – now called a virtual fundamental class – in the Čech homology of *every* fiber of $\overline{\pi}$. We then give a precise definition of virtual fundamental class (Definition 3.4) and prove:

Theorem. *Every thinly compactified family $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ admits a unique virtual fundamental class.*

Section 4 describes several ways in which a virtual fundamental class on one thinly compactified family induces virtual fundamental classes on related families.

Section 5 applies these ideas to Donaldson theory. Given an oriented Riemannian manifold (X, g), one associates moduli spaces $\mathcal{M}_k(g)$ of g-anti-self-dual $U(2)$-connections. Donaldson's polynomial invariants are defined by evaluating certain natural cohomology classes on $\mathcal{M}_k(g)$ for a generic g. We show that results already present in Donaldson's work imply the existence of virtual fundamental classes for the Uhlenbeck compactification $\overline{\mathcal{M}}_k(g)$ for *any* metric.

Sections 6-10 give applications to Gromov-Witten theory. Here the central objects are the moduli spaces of stable maps into a closed symplectic manifold (X, ω), viewed as a family

$$\mathcal{M}_{A,g,n}(X) \to \mathcal{JV} \tag{0.2}$$

over the space of Ruan-Tian perturbations (described in Section 6). Again, the theme is that many results in the literature can be viewed as giving conditions under which there exist thin compactifications of the universal Gromov-Witten moduli spaces (0.2) over $\mathcal{JV}$ or over some subset of $\mathcal{JV}$. The results of Sections 2–4 then immediately imply the existence of a virtual fundamental class on the fibers of the moduli space over the same subset of $\mathcal{JV}$.

As examples, we discuss moduli spaces of somewhere-injective maps in Section 7, of domain-fine maps in Section 8, and of relative domain-fine maps in Section 9. In each case, one obtains a virtual fundamental class in Čech homology over a subset of $\mathcal{JV}$. Under special assumptions on the data (X, A, g, n), this is true over all of $\mathcal{JV}$, in which case the virtual fundamental class determines Gromov-Witten numbers that are invariants of the

symplectic structure of X. In Section 10 we use this viewpoint to briefly describe how virtual fundamental classes exist in two general cases: for semipositive manifolds (X, ω), as done by Ruan and Tian [RT1, RT2], and the genus $g = 0$ case, as done by Cielieback and Mohnke [CM].

John Pardon recently constructed a virtual fundamental class on the space of stable maps for any genus and any closed symplectic manifold [Pd1]. He also uses Čech theory, but his approach is different. He considers the space of stable maps $\overline{\mathcal{M}^J}$ for a fixed almost complex structure J, and constructs perturbations ν_α of the J-holomorphic map equation on open sets U_α that cover $\overline{\mathcal{M}^J}$, and shows that resulting perturbations of the U_α define a cycle in the dual of Čech cohomology. Our approach should be compatible with his, although we do not attempt to make this precise.

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1. Steenrod and Čech homologies

Expositions of Steenrod homology are surprisingly hard to find in the literature. We will use the version of Steenrod homology that is based on "infinite chains", as presented in Chapter 4 of W. Massey's book [Ma2]. To avoid ambiguity, we denote this theory by ${}^s\mathrm{H}_*$. For background, see also [Ma1], [Mil] and the introduction to [Ma2].

Steenrod homology with coefficient group G assigns, for each integer p, an abelian group ${}^s\mathrm{H}_p(X) = {}^s\mathrm{H}_p(X, G)$ to each locally compact Hausdorff space X, and a homomorphism $f_* : {}^s\mathrm{H}_p(X) \to {}^s\mathrm{H}_p(Y)$ to each *proper* continuous map. The axioms for this homology theory [Ma2, p. 86] include:

- For each open subset $U \subset X$ and each p, there is a natural "restriction" map
$$\rho_{X,U} : {}^s\mathrm{H}_p(X) \to {}^s\mathrm{H}_p(U). \tag{1.1}$$
- For each closed set $\iota : A \hookrightarrow X$, there is a natural long exact sequence
$$\cdots \longrightarrow {}^s\mathrm{H}_p(A) \xrightarrow{i_*} {}^s\mathrm{H}_p(X) \xrightarrow{\rho} {}^s\mathrm{H}_p(X - A) \xrightarrow{\partial} {}^s\mathrm{H}_{p-1}(A) \longrightarrow \cdots \tag{1.2}$$
- If X is the union of disjoint open subsets $\{X_\alpha\}$, then the inclusions $\iota_\alpha : X_\alpha \to X$ induce monomorphisms in homology, and ${}^s\mathrm{H}_p(X)$ is the cartesian product
$${}^s\mathrm{H}_p(X) \;=\; \prod_\alpha \, (\iota_\alpha)_* {}^s\mathrm{H}_p(X_\alpha). \tag{1.3}$$
- For any inverse system $\{\cdots \to Y_3 \to Y_2 \to Y_1\}$ of compact metric spaces with limit Y, the maps $Y \to Y_\alpha$ induce a natural exact sequence [Mil, Theorem 4]
$$0 \longrightarrow \lim{}^1 \left[{}^s\mathrm{H}_{p+1}(Y_\alpha; G)\right] \longrightarrow {}^s\mathrm{H}_p(Y; G) \longrightarrow \varprojlim {}^s\mathrm{H}_p(Y_\alpha; G) \longrightarrow 0. \tag{1.4}$$

The corresponding cohomology theory is Alexander-Spanier cohomology with compact support. For compact Haudorff spaces, this is isomorphic to both Alexander-Spanier and Čech cohomology $\check{\mathrm{H}}^*$ [Sp, p. 334], and there is a universal coefficient theorem ([Ma2], Cor. 4.18)
$$0 \longrightarrow \mathrm{Ext}(\check{\mathrm{H}}^{d+1}(\overline{M}, G), G) \longrightarrow {}^s\mathrm{H}_d(\overline{M}, G) \longrightarrow \mathrm{Hom}(\check{\mathrm{H}}^d(\overline{M}), G) \longrightarrow 0. \tag{1.5}$$

One also has the following facts about oriented topological manifolds M of dimension d (not necessarily compact) and any coefficient group:

- For all $p > d$,
$$ {}^s\mathrm{H}^p(M) = 0 \qquad \text{and} \qquad {}^s\mathrm{H}_p(M) = 0. \tag{1.6}$$
- For each topological d-ball B in a connected component M_i of M, ${}^s\mathrm{H}_d(B;G) \cong G$ and
$$ \rho_{M_i,B} : {}^s\mathrm{H}_d(M_i) \to {}^s\mathrm{H}_d(B) \qquad \text{is an isomorphism.} \tag{1.7}$$
- The orientation determines a fundamental class $[M] \in {}^s\mathrm{H}_d(M)$. If M has components M_α, the fundamental class is given under the isomorphism (1.3) by
$$ [M] \;=\; \prod_\alpha [M_\alpha]. \tag{1.8}$$

For proofs, see [Ma2], Theorems 2.13 and 3.21a and page 112.

In Section 2, we work exclusively with Steenrod homology. In Section 3, where we consider families of spaces, we pass instead to Čech homology, because it satisfies the following

Continuity Property. For every inverse system of compact Hausdorff spaces as in (1.4), the maps $Y \to Y_\alpha$ induce a natural isomorphism
$$ \check{\mathrm{H}}_*(Y;G) \xrightarrow{\cong} \varprojlim \check{\mathrm{H}}_*(Y_\alpha;G) \tag{1.9}$$
[ES, pages 260-261].

In general, Steenrod homology does not satisfy the continuity property (it satisfies (1.4) instead), and Čech homology does not satisfy the exactness axiom. However, for every compact Hausdorff space X and any coefficient group G, there are natural maps
$$ {}^s\mathrm{H}_p(X;G) \longrightarrow \check{\mathrm{H}}_p(X;G) \longrightarrow \check{\mathrm{H}}^p(X;G)^\vee \tag{1.10}$$
where $\check{\mathrm{H}}^p(X;G)^\vee = \mathrm{Hom}(\check{\mathrm{H}}^p(X;G), G)$ is the dual to Čech cohomology (cf. Remark 5.0.3 in [Pd1]). Furthermore, when restricted to compact metric spaces and rational coefficients, both arrows in (1.10) are isomorphisms (the first arrow by Milnor's uniqueness theorem [Mil]), giving a theory that is both exact and continuous (cf. [ES, p. 233]).

Lemma 1.1. *Let $\mathcal{H}(X)$ denote one of the three possibilities:*
$$ \mathcal{H}(X) \;=\; \begin{cases} \check{\mathrm{H}}_*(X;\mathbb{Z}) & \text{Čech homology, or} \\ \check{\mathrm{H}}^*(X;\mathbb{Z})^\vee & \text{Dual Čech cohomology, or} \\ \check{\mathrm{H}}_*(X;\mathbb{Q}) & \text{Rational Čech homology.} \end{cases} \tag{1.11}$$
Then there is a natural map ${}^s\mathrm{H}_*(X;\mathbb{Z}) \to \mathcal{H}(X)$ defined on the category of compact Hausdorff spaces, and $\mathcal{H}$ satisfies the Continuity Property (i.e. (1.9) *holds with $\check{\mathrm{H}}_*$ replaced by $\mathcal{H}$).*

Proof. For any coefficient module G, Čech homology satisfies (1.9) while, with the same notation, Čech cohomology satisfies
$$ \check{\mathrm{H}}^p(Y,G) = \varinjlim_\alpha \check{\mathrm{H}}^p(Y_\alpha, G) \tag{1.12}$$
[ES, pages 260-261]. Hence by Proposition 5.26 in [Ro],
$$ \check{\mathrm{H}}^p(Y,G)^\vee = \mathrm{Hom}(\varinjlim_\alpha \check{\mathrm{H}}^p(Y_\alpha,G), G)) = \varprojlim_\alpha \mathrm{Hom}(\check{\mathrm{H}}^p(Y_\alpha,G), G) = \varprojlim_\alpha \check{\mathrm{H}}^p(Y_\alpha,G)^\vee. $$
□

Each of the possibilities in Lemma 1.1 pairs with Čech cohomology; there is no longer any need for Alexander-Spanier cohomology. Čech cohomology, of course, is different from singular cohomology but, for any G and any paracompact Hausdorff space X, there is a natural map

$$\check{\mathrm{H}}^p(X;G) \to H^p_{sing}(X;G) \tag{1.13}$$

that is an isomorphism if X is a manifold, or more generally if X is locally contractible [Sp, Corollaries 6.8.8 and 6.9.5].

2. Thin compactifications

In Steenrod homology with integer coefficients, open manifolds M have a fundamental class, but this class is of limited use because it does not push forward under general continuous maps. This deficiency can be rectified by considering maps that extend continuously over a compactification $\overline{M} = M \cup S$ of M, and showing that $\overline{M}$ carries a fundamental class. Many such compactifications are possible; making S larger allows more maps to extend continuously to $\overline{M}$, but making S too large interferes with the fundamental class. Definition 2.1 identifies a class of compactifications – "thin compactifications" – that is appropriate for working with fundamental classes. These have the form

$$\overline{M} = M \cup S$$

where S is a space of (homological) codimension 2. There are no assumptions about differentiability or about how M and S fit together, other than the fact that the total space is a compact Hausdorff space.

Definition 2.1. *Let M be an oriented d-dimensional topological manifold. A* thin compactification *of M is a compact Hausdorff space $\overline{M}$ containing M such that the complement $S = \overline{M} \smallsetminus M$ (the "singular locus") is a closed subset of codimension 2 in the sense that*

$$^s\mathrm{H}_p(S) = 0 \qquad \forall\, p > d-2. \tag{2.1}$$

Every compact manifold is a thin compactification (with S empty), and for each manifold of finite dimension $d \geq 2$, the 1-point compactification is a thin compactification. Further examples arise by applying the following lemma communicated to us by both J. Morgan and J. Pardon.

Lemma 2.2. *Suppose that a compact Haudorff space S is a union of closed sets S_i, $i \geq 0$, such that for each i, $S_{i+1} \subset S_i$ and $S_i - S_{i+1}$ is a manifold of dimension at most $d-i$ or more generally has Steenrod homological dimension $d-i$. Then S has homological dimension at most d, i.e. $^s\mathrm{H}_k(S) = 0$ for all $k > d$.*

Proof. By induction on i, starting from $i = d$, one can assume that $^s\mathrm{H}_q(S_i) = 0$ for all $q > d-i$. This is because the long exact sequence

$$\to {}^s\mathrm{H}_q(S_{i+1}) \to {}^s\mathrm{H}_q(S_i) \to {}^s\mathrm{H}_q(S_i - S_{i+1}) \to {}^s\mathrm{H}_{q-1}(S_{i+1}) \to$$

and the induction assumption implies $^s\mathrm{H}_q(S_i) \xrightarrow{\cong} {}^s\mathrm{H}_q(S_i - S_{i+1})$ for all $q > d-i-1$. But $^s\mathrm{H}_q(S_i - S_{i+1}) = 0$ for all $q > d-i$ and therefore $^s\mathrm{H}_q(S_i) = 0$ for all $q > d-i$. □

In practice, singular strata are usually unions of a large number of strata S_α. One must form the S_i of Lemma 2.2 as unions of the S_α and verify that $S_i - S_{i-1}$ are manifolds. One way of doing this is described in the appendix.

Examples.

(a) The closure $\overline{V}$ of a smooth quasi-projective variety $V \subset \mathbf{P}^N$ is a thin compactification.

(b) For a nodal complex curve C, the regular part $M = C^{reg}$ can be thinly compactified in three ways: by its 1-point compactification, by C, and by its normalization $\tilde{C}$, which may be disconnected.

(c) Define an infinite chain of 2-spheres as follows. For each $n = 1, 2, \ldots$, let p_n be the point $(\frac{1}{n}, 0, 0)$ in $\mathbb{R}^3$. Let S_n be the sphere with center $q_n = \frac{1}{2}(p_n + p_{n+1})$ and radius $R_n = |p_n - q_n|$ with the two points p_n and p_{n+1} removed. Then $M = \bigcup S_n$ is an embedded 2-manifold in $\mathbb{R}^3$, and $\overline{M} = M \cup S$ is a thin compactification with a singular set $S = \bigcup p_n \cup (0,0,0)$ of dimension zero.

(d) In contrast, $M = \{\frac{1}{n} \,|\, n \in \mathbb{Z}\} \subset \mathbb{R}$ is a 0-manifold, but its compactification $M \cup \{0\}$ is not thin.

Theorem 2.3. *Let M be an oriented d-dimensional manifold with fundamental class $[M]$. Every thin compactification $\overline{M}$ of M has a fundamental class*

$$[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M}; \mathbb{Z})$$

uniquely characterized by the requirement that

$$\rho_M([\overline{M}]) = [M], \tag{2.2}$$

where $\rho_M : {}^s\mathrm{H}_d(\overline{M}; \mathbb{Z}) \to {}^s\mathrm{H}_d(M; \mathbb{Z})$ is the map (1.1).

Proof. The exact sequence (1.2) for the closed subset $A = S$ of $\overline{M}$, together with (2.1), implies that the map

$$\rho_M : {}^s\mathrm{H}_\ell(\overline{M}) \xrightarrow{\cong} {}^s\mathrm{H}_\ell(M) \tag{2.3}$$

is an isomorphism for all $\ell \geq d$. Taking $\ell = d$ shows that there is a unique class $[\overline{M}]$ satisfying (2.2). □

In general, a manifold M has many thin compactifications, each with a fundamental class related to $[M]$ by (2.2). If $\overline{M}$ is one such thin compactification with singular locus S, and $Z \subset M$ is a closed subset such that $Z \cup S$ has homological codimension 2, then $\overline{M}$ is also a thin compactification of $M \smallsetminus Z$, and $[\overline{M \smallsetminus Z}] = [\overline{M}]$. In this sense, one can ignore sets of codimension 2 in computations with fundamental classes.

Example 2.4. For two thin compactifications $\overline{M}_1$ and $\overline{M}_2$ of the same d-dimensional manifold M, there are isomorphisms $\rho_i : {}^s\mathrm{H}_d(\overline{M}_i) \to {}^s\mathrm{H}_d(M)$, as in (2.3), and the composition

$$\rho_2^{-1} \circ \rho_1 : {}^s\mathrm{H}_d(\overline{M}_1) \to {}^s\mathrm{H}_d(\overline{M}_2)$$

takes $[\overline{M}_1]$ to $[\overline{M}_2]$. This is true even when there is no continuous map from $\overline{M}_1$ to $\overline{M}_2$. If there is a map $f : \overline{M}_1 \to \overline{M}_2$, then $f_*[\overline{M}_1] = [\overline{M}_2]$ by the naturality of ρ. In particular:

(a) Let $\pi : M_Z \to M$ be the blowup of a closed complex manifold M along a complex submanifold Z. Then M and M_Z are two different thin compactifications of $M \smallsetminus Z$, and $\pi_*[M_Z] = [M]$.

(b) More generally, a rational map $X \dashrightarrow Y$ between complex projective varieties induces an identification of $[X]$ with $[Y]$.
(c) If $\dim M \geq 2$, every thin compactification $\overline{M}$ has a map p to the 1-point compactification M^+, and $p_*[\overline{M}] = [M^+]$.

The fundamental class of a manifold M need not push forward under a general continuous map $f : M \to X$. However, if f extends to a continuous map $\overline{f} : \overline{M} \to X$ from *some* thin compactification $\overline{M}$ of M, then $\overline{f}$ is proper, so induces a map $\overline{f}_*$ in Steenrod homology:

$$\begin{array}{ccc} {}^s\mathrm{H}_d(\overline{M}) & & \\ \rho \downarrow \cong & \searrow^{\overline{f}_*} & \\ {}^s\mathrm{H}_d(M) & \dashrightarrow & {}^s\mathrm{H}_d(X). \end{array}$$

Then $[M]$ corresponds to $[\overline{M}]$ by (2.2), and the class $\overline{f}_*([\overline{M}]) \in {}^s\mathrm{H}_d(X)$ serves as a surrogate for $f_*[M]$. Alternatively, one can take a Čech class $\alpha \in \check{\mathrm{H}}^d(X)$ and evaluate $\overline{f}^*\alpha$ on the image of $[\overline{M}]$ under (1.10).

2.1. Covering maps. The isomorphism (2.3) implies several statements about how fundamental classes behave under covering maps.

Lemma 2.5. *Suppose that $\overline{f} : \overline{M} \to \overline{N}$ is a continuous map between thinly compactified oriented manifolds that restricts to a finite oriented covering $f : M \to N$. If f has degree ℓ, then*

$$\overline{f}_*[\overline{M}] = \ell\,[\overline{N}]. \tag{2.4}$$

More generally, if N has components $\{N_\alpha\}$ then, in the notation of (1.3) *and* (1.8),

$$\overline{f}_*[\overline{M}] \;=\; \prod_\alpha \ell_\alpha\,[\overline{N}_\alpha] \tag{2.5}$$

where ℓ_α is the degree of the restriction of f to $f^{-1}(N_\alpha)$ (and 0 if this set is empty).

Proof. First assume that M and N are both connected. Fix an open ball $U \subset N$ so that $f^{-1}(U)$ is the disjoint union of ℓ open balls $V_1, \dots V_\ell$. In this situation, there is an isomorphism $\rho_U : {}^s\mathrm{H}_d(N) \to {}^s\mathrm{H}_d(U)$ as in (1.7), and similar isomorphisms $\rho_i : {}^s\mathrm{H}_d(M) \to {}^s\mathrm{H}_d(V_i)$ for each i. These fit into a commutative diagram

$$\begin{array}{ccccccc} {}^s\mathrm{H}_d(\overline{M}) & \xrightarrow[\rho_M]{\cong} & {}^s\mathrm{H}_d(M) & \xrightarrow[(\rho_1,\dots,\rho_\ell)]{} & \bigoplus_i {}^s\mathrm{H}_d(V_i) & \xrightarrow{\cong} & \mathbb{Z}\oplus\cdots\oplus\mathbb{Z} \\ \overline{f}_*\downarrow & & f_*\downarrow & & f_*\downarrow & & \varphi\downarrow \\ {}^s\mathrm{H}_d(\overline{N}) & \xrightarrow[\rho_N]{\cong} & {}^s\mathrm{H}_d(N) & \xrightarrow[\rho_U]{\cong} & {}^s\mathrm{H}_d(U) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

where $\varphi(a_1, \dots, a_\ell) = \sum a_i$, where ρ_M and ρ_N are isomorphisms by (2.3), and where the first two squares commute by the naturality of ρ. Restricting the diagram to generators gives (2.4).

In general, for each component N_α of N, $f^{-1}(N_\alpha)$ is the disjoint union of components $M_{\alpha\beta}$, and (2.4) applies to each restriction $f_{\alpha\beta} = f|_{M_{\alpha\beta}}$, and the homologies of $\overline{M}$ and $\overline{N}$ are cartesian products as in (1.3). This implies (2.5) with $\ell_\alpha = \sum_\beta \deg f_{\alpha\beta}$, and (2.4) if all ℓ_α are equal to ℓ. □

Example 2.6. Lemma 2.5 applies to branched covers of complex analytic varieties.

2.2. Components. Suppose that an oriented manifold M has finitely many connected components M_α, and that $\overline{M}$ is a thin compactification of M with singular locus S. We then have:

Lemma 2.7. *For each α, $\overline{M}_\alpha = M_\alpha \cup S$ is a thin compactification of M_α, and*

$$[\overline{M}] = \sum_\alpha [\overline{M}_\alpha]. \tag{2.6}$$

Proof. The first statement holds because $\overline{M}_\alpha = M_\alpha \cup S$ is a closed, hence compact, subset of $\overline{M}$ and S satisfies (2.1). The disjoint union $\sqcup \overline{M}_\alpha$ is therefore another thin compactification of M, and $[\sqcup \overline{M}_\alpha] = \sum_\alpha [\overline{M}_\alpha]$. Moreover, the identity $M \to M$ extends to a continuous map $\iota : \sqcup \overline{M}_\alpha \to \overline{M}$. Lemma 2.5 then gives $\iota_*[\sqcup \overline{M}_\alpha] = [\overline{M}]$, and hence (2.6). □

2.3. Thin Compactifications with boundary. It is useful to extend the notion of thin compactifications to manifolds M with boundary ∂M.

Definition 2.8. *A* thin compactification *of $(M, \partial M)$ is a compact Hausdorff pair $(\overline{M}, \overline{\partial M})$ containing $(M, \partial M)$ such that*

(i) $S = \overline{M} \smallsetminus M$ is a closed subset of $\overline{M}$ of codimension 2,
(ii) $S' = \overline{\partial M} \smallsetminus \partial M$ is a closed subset of $\overline{\partial M}$ of codimension 2, and
(iii) $S' \subseteq S$.

Note that (ii) implies that $\overline{\partial M}$ is a thin compactification of ∂M, while (iii) implies that the interior $M^0 = M \smallsetminus \partial M$ is a subset of $\overline{M} \smallsetminus \overline{\partial M}$. The exact sequence (1.2) of such a pair $(\overline{\partial M}, \overline{M})$ is, in part,

$$ {}^s\mathrm{H}_d(\overline{M}) \xrightarrow{\rho} {}^s\mathrm{H}_d(\overline{M} \smallsetminus \overline{\partial M}) \xrightarrow{\partial} {}^s\mathrm{H}_{d-1}(\overline{\partial M}) \xrightarrow{\iota_*} {}^s\mathrm{H}_{d-1}(\overline{M}). \tag{2.7}$$

When M is oriented, there is an induced orientation on ∂M, and the interior M^0 carries a fundamental class $[M^0] \in H_d(M^0)$. This is related to the fundamental class $[\partial M]$ of ∂M by

$$\partial [M^0] = [\partial M] \in {}^s\mathrm{H}_{d-1}(\partial M) \tag{2.8}$$

where ∂ is the boundary operator in the sequence (1.2) for the pair $(M, \partial M)$ (see [Ma2, Theorem 11.8], being mindful of orientations and noting the change of notation $H_p \mapsto H_p^\infty$ on page 302).

Lemma 2.9. *A thin compactification $(\overline{M}, \overline{\partial M})$ of an oriented d-dimensional manifold-with-boundary $(M, \partial M)$ has a natural fundamental class $[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M} \smallsetminus \overline{\partial M})$ such that, for the maps in* (2.7)*,*

$$(a)\ \partial[\overline{M}] = [\overline{\partial M}] \qquad \text{and} \qquad (b)\ \bar{\iota}_*[\overline{\partial M}] = 0. \tag{2.9}$$

Furthermore, $\rho'[\overline{M}] = [M^0]$ under the restriction to $M^0 \subseteq \overline{M} \smallsetminus \overline{\partial M}$.

Proof. Combining (2.7) with the similar sequence for the pair $(M, \partial M)$ gives the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}^s\mathrm{H}_d(\overline{M}) & \xrightarrow{\rho} & {}^s\mathrm{H}_d(\overline{M} \smallsetminus \overline{\partial M}) & \xrightarrow{\partial} & {}^s\mathrm{H}_{d-1}(\overline{\partial M}) & \xrightarrow{\bar{\iota}_*} & {}^s\mathrm{H}_{d-1}(\overline{M}) \\
 & & \downarrow{\scriptstyle \rho_{\overline{M},M}} & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho_\partial} & & \downarrow{\scriptstyle \rho} \\
0 & \longrightarrow & {}^s\mathrm{H}_d(M) & \xrightarrow{\rho} & {}^s\mathrm{H}_d(M^0) & \xrightarrow{\partial} & {}^s\mathrm{H}_{d-1}(\partial M) & \xrightarrow{\iota_*} & {}^s\mathrm{H}_{d-1}(M)
\end{array}$$

where the rows are exact and the vertical maps are restriction maps to open subsets. Using properties 3b, 4b, and 4c listed on page 86 of [Ma2], one sees that the three squares are commutative. The first and third vertical arrows are isomorphisms by parts (i) and (ii) of Definition 2.8, and the exact sequence (1.2) for the pair $(\overline{M}, S)$ shows that ρ is an injection. The Five Lemma then implies that ρ' is an isomorphism.

We can define $[\overline{M}] \in {}^s\mathrm{H}_d(\overline{M} \smallsetminus \overline{\partial M})$ uniquely by the requirement

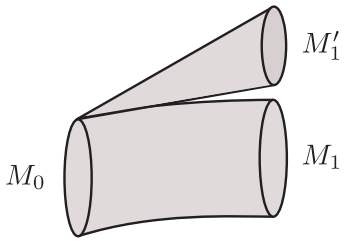
$$\rho'[\overline{M}] = [M^0]$$

Then (2.9a) follows from (2.8) and the uniqueness of (2.2), while (2.9b) follows from exactness of the top row of the diagram. □

Examples.

(a) If X is a manifold of dimension $d \geq 1$ with thin compactification $\overline{X}$, then the cone $\overline{M}$ on $\overline{X}$ is a thin compactification of M, the cone on X minus the vertex.

(b) In the picture, $\overline{M}$ is the union of a cone on S^2 and a cylinder $S^2 \times [0,1]$, intersecting a one point p. Then the complement of the cone point p is a manifold with boundary, and $\overline{M}$ satisfies the conditions of Definition 2.8 with $S = S' = \{p\}$.



2.4. Cobordisms. Lemma 2.9 can be applied to cobordisms.

Corollary 2.10. *Suppose that W is an oriented topological cobordism between d-dimensional manifolds M_0 and M_1. If W admits a thin compactification $(\overline{W}, \overline{\partial W})$, then the fundamental classes of $\overline{M}_0$ and $\overline{M}_1$ represent the same class in $\overline{W}$.*

Proof. The hypothesis means that W is an oriented topological manifold with boundary $M_1 \sqcup -M_0$. Then Lemmas 2.7 and 2.9 applies, and (2.9b) becomes the statement

$$(\iota_0)_*[\overline{M}_0] = (\iota_1)_*[\overline{M}_1] \qquad \text{in } {}^s\mathrm{H}_d(\overline{W}) \tag{2.10}$$

where ι_0, ι_1 are the inclusions of $\overline{M}_0$ and $\overline{M}_1$ into $\overline{W}$. □

3. Virtual fundamental classes of families

The notion of thin compactification extends to families of manifolds, where the fundamental class is replaced by the "virtual fundamental class" of the fibers. In this section the word "manifold" means a smooth separable Banach manifold, finite or infinite dimensional. As usual, the term "second category" means a countable intersection of open dense subsets, and we say that a property holds "for generic p" if it holds for all p in some second category set.

We will consider Fredholm maps

$$\begin{array}{c} \mathcal{M} \\ \big\downarrow \pi \\ \mathcal{P} \end{array} \tag{3.1}$$

between manifolds, which we regard as a family of spaces (the fibers of π) parameterized by $\mathcal{P}$. Such a map has an associated Fredholm index d which, by the following well-known theorem, is the dimension of the generic fibers of π.

Theorem 3.1 (Sard-Smale [S])**.** *For a Fredholm map* (3.1) *of index d,*

(a) The set $\mathcal{P}^* \subset \mathcal{P}$ of regular values of π is a second category set, and for each $p \in \mathcal{P}^*$, the fiber $\pi^{-1}(p)$ is a manifold of dimension d, and is empty if $d < 0$.

(b) For any $p, q \in \mathcal{P}^*$, every smooth path $[0,1] \to \mathcal{P}$ from p to q is the C^0 limit of paths $\sigma_k : [0,1] \to \mathcal{P}$ from p to q such that $\pi_k^{-1}(\sigma)$ is manifold of dimension $d+1$.

The data (3.1) also determines a real line bundle $\det d\pi$ over $\mathcal{M}$ – the determinant bundle of the Fredholm map π— whose restriction to each regular fiber $\mathcal{M}_p = \pi^{-1}(p)$, $p \in \mathcal{P}$, is the orientation bundle $\Lambda^d T^*\mathcal{M}_p$. We will always assume that (3.1) has a *relative orientation* specified by a nowhere zero section of $\det d\pi$.

The definition of a thin compactification for families is designed so that regular fibers are thinly compactified in the sense of Definition 2.1

Definition 3.2. *A* thin compactification *of a relatively oriented family* (3.1) *is a Hausdorff space $\overline{\mathcal{M}}$ containing $\mathcal{M}$ as an open subset, and an extension of π to a proper continuous map $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ such that $\overline{\mathcal{M}}$ is a disjoint union*

$$\overline{\mathcal{M}} = \mathcal{M} \cup \bigcup_{k=2}^{\infty} \mathcal{S}_k \tag{3.2}$$

where each $\mathcal{S}_k$ is a manifold and each restriction $\pi_k = \overline{\pi}|_{\mathcal{S}_k}$ is a Fredholm map $\pi_k : \mathcal{S}_k \to \mathcal{P}$ of index $d-k$.

We call $\mathcal{S}_k$ the *codimension k stratum* of $\overline{\mathcal{M}}$. As in Section 2, the adjective "thin" is meant to emphasize that all boundary strata $\mathcal{S}_k$ in (3.2) have codimension 2 or more.

Given such a thin compactification, we can apply the Sard-Smale Theorem to (3.1) and to each map $\pi_k : \mathcal{S}_k \to \mathcal{P}$ and intersect the resulting second category sets. The result is a single second category set $\mathcal{P}^* \subset \mathcal{P}$ whose points are simultaneous regular values of π and all π_k; we call these regular values of $\overline{\pi}$. The fiber $\overline{\mathcal{M}}_p$ of $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ over each $p \in \mathcal{P}^*$ then satisfies the hypotheses of Lemma 2.2, so is a thin compactification of the manifold $\mathcal{M}_p = \pi^{-1}(p)$. By Theorem 2.3, each regular fiber therefore carries a fundamental class in Steenrod homology

$$[\overline{\mathcal{M}}_p] \in {}^s\mathrm{H}_d(\overline{\mathcal{M}}_p; \mathbb{Z}). \tag{3.3}$$

Similarly, for any $p, q \in \mathcal{P}^*$, there is a dense set in the space of paths γ in $\mathcal{P}$ from p to q such that each γ in this dense set is transverse to π_k for all k. Then the Sard-Smale Theorem implies that $\overline{\mathcal{M}_\gamma} = \overline{\pi}^{-1}(\gamma)$ is a thin compactification of $\mathcal{M}_\gamma$, and hence by Corollary 2.10 the images under the endpoint inclusions ι_p and ι_q are equal in the homology of $\overline{\mathcal{M}}_\gamma$:

$$(\iota_p)_*[\overline{\mathcal{M}}_p] \;=\; (\iota_q)_*[\overline{\mathcal{M}}_q] \in {}^s\mathrm{H}_d(\overline{\mathcal{M}}_\gamma, \mathbb{Z}). \tag{3.4}$$

We now pass from Steenrod to Čech homology. By Lemma 1.1, for each $p \in \mathcal{P}^*$, the fundamental class (3.3) defines a class

$$[\overline{\mathcal{M}}_p] \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_p), \tag{3.5}$$

in integral Čech homology, and (3.4) continues to hold in $\check{\mathrm{H}}_d(\overline{\mathcal{M}}_\gamma)$ for all $p, q \in \mathcal{P}^*$. In this setting, the association $p \mapsto [\overline{\mathcal{M}}_p]$ now extends to all $p \in \mathcal{P}$ by the continuity property (1.9), in the following form.

Lemma 3.3 (Extension Lemma)**.** *Assume $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ is a proper continuous map from a Hausdorff space to a separable Banach manifold. Suppose that there is a dense subset $\mathcal{P}^*$ of $\mathcal{P}$ and a map*

$$p \mapsto \alpha_p \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_p) \tag{3.6}$$

defined for $p \in \mathcal{P}^*$ such that, for all $p, q \in \mathcal{P}^*$, there exists a dense set of piecewise smooth paths γ from p to q for which the endpoint inclusions induce an equality

$$\iota_* \alpha_p = \iota_* \alpha_q \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_\gamma). \tag{3.7}$$

Then (3.6) *extends to all $p \in \mathcal{P}$ so that* (3.7) *holds for all paths γ from p to q, and this extension is unique.*

Proof. Fix $p \in \mathcal{P}$ and identify a neighborhood of p with a neighborhood of the origin in a Banach space. The balls B_k of radius $1/k$ centered at p each contain a dense set of values $q \in \mathcal{P}^* \cap B_k$ for which $\alpha_q \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_q)$ is defined. Moreover, any two values in $\mathcal{P}^* \cap B_k$ can be joined by a line segment in B_k which, by assumption, can be perturbed to a path in B_k for which (3.7) holds.

Choose any sequence $p_k \in B_k \cap \mathcal{P}^*$ converging to p and paths $\gamma_k \subset B_k$ from p_k to p_{k+1} satisfying (3.7). For each m, the set

$$K_m \;=\; \{p\} \;\cup\; \bigcup_{k \geq m} \gamma_k$$

is compact, and $\overline{\mathcal{M}}_m = \pi^{-1}(K_m)$ is a sequence of nested compact Hausdorff spaces whose intersection is the compact space $\overline{\mathcal{M}}_p$.

For each $k \geq m$, the images under the inclusions $\overline{\mathcal{M}}_{p_k} \hookrightarrow \overline{\mathcal{M}}_m$ determine a homology class

$$\alpha_{p_k} \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_m) \tag{3.8}$$

which, by (3.7), is independent of k. These homology classes are consistently related by the inclusions $\overline{\mathcal{M}}_{m_1} \hookrightarrow \overline{\mathcal{M}}_{m_2}$ for $m_1 \geq m_2$, so define an element of the inverse system

$$\varprojlim_m \alpha_{p_k} \in \varprojlim_m \check{\mathrm{H}}_d(\overline{\mathcal{M}}_m). \tag{3.9}$$

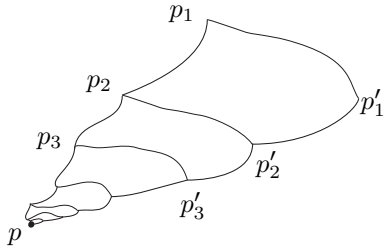
By the continuity property (1.9), this determines a unique Čech homology class

$$\alpha_p \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_p). \tag{3.10}$$

If K'_m is another such broken path, we can similarly find paths σ_k between p_k and p'_k inside B_k, as shown in the figure, for which (3.7) holds. By (3.7), for each $k \le m$, the images of α_{p_k} and $\alpha_{p'_k}$ are equal in the homology of the compact Hausdorff space $\overline{\mathcal{M}}_{L_m} = \pi^{-1}(L_m)$ where L_m is the "ladder"

$$L_m = K_m \cup K'_m \cup \{\sigma_k \,|\, k \ge m\}.$$

so the element (3.9) of the inverse system, and hence the limit (3.10), is well-defined for every $p \in \mathcal{P}$, independent of choices.



Similarly, to verify relation (3.7) for any path γ in $\mathcal{P}$, first choose broken paths K_m and K'_m as above that limit to the endpoints p and p' of γ, respectively. For each k, we can choose a path σ_k between p_k and p'_k that lies in the $1/k$ neighborhood of γ, and for which (3.7) holds. Then the ladders L_m, defined as above, are a nested sequence of compacta converging to γ such that, after including into L_m,

$$\alpha_p = \alpha_{p_k} = \alpha_{p'_k} = \alpha_{p'} \in \check{\mathrm{H}}_*(\overline{\mathcal{M}}_{L_m})$$

for all $k \ge m$. Again applying the continuity property, we conclude that $\alpha_p = \alpha_{p'} \in \check{\mathrm{H}}_*(\overline{\mathcal{M}}_\gamma)$.

Finally, to check uniqueness, assume α' is another extension which agrees with α on $\mathcal{P}^*$ and satisfies (3.7) for all paths γ in $\mathcal{P}$. Pick any point $p \in \mathcal{P}$ and broken paths K_m as above. Then for any $k \ge m$, the inclusions induce equalities

$$\alpha'_p = \alpha'_{p_k} = \alpha_{p_k} \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_{K_m}).$$

Therefore, again by continuity,

$$\alpha'_p = \varprojlim_m \alpha'_p = \varprojlim_m \alpha_{p_k} = \alpha_p$$

in $\varprojlim \check{\mathrm{H}}_d(\overline{\mathcal{M}}_{K_m}) = \check{\mathrm{H}}_d(\overline{\mathcal{M}}_p)$. This completes the proof. □

We can now give an axiomatic definition of virtual fundamental classes for thinly compactified families. Let $\check{\mathrm{H}}_*$ denote any one of the possibilities in (1.11).

Definition 3.4. A *virtual fundamental class* (VFC) of a thinly compactified family $\overline{\mathcal{M}} \to \mathcal{P}$ of index d associates to each compact path connected subset $Q \subset \mathcal{P}$ an element

$$[\overline{\mathcal{M}}_Q]^{vir} \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}_Q) \tag{3.11}$$

such that:

A1. For each regular $p \in \mathcal{P}$, $[\overline{\mathcal{M}}_p]^{vir}$ is the fundamental class $[\overline{\mathcal{M}}_p]$.

A2. Every inclusion $\iota : R \to Q$ of compact path connected subsets induces an equality $\iota_*[\overline{\mathcal{M}}_R]^{vir} = [\overline{\mathcal{M}}_Q]^{vir}$.

The virtual fundamental class (3.11) is called integral if it lies in $\check{\mathrm{H}}_d(\overline{\mathcal{M}}_Q;\mathbb{Z})$ and rational if it lies in $\check{\mathrm{H}}_d(\overline{\mathcal{M}}_Q;\mathbb{Q})$. As in Lemma 1.1, an integral and rational virtual fundamental classes both determine a class in $\check{\mathrm{H}}^d(\overline{\mathcal{M}}_Q;\mathbb{Q})^\vee$, where Pardon's virtual fundamental classes are defined [Pd1].

With this terminology, our main result can be stated simply:

Theorem 3.5. *Every thinly compactified family $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ admits a unique virtual fundamental class.*

Proof. Let $\mathcal{P}^*$ be the dense set of regular values of $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$. The discussion containing (3.3) and (3.4) shows that, after passing to Čech homology, the map $p \mapsto [\overline{\mathcal{M}}_p]$ is defined on $\mathcal{P}^*$, and satisfies (3.7) for the dense set of paths described above (3.4). By Lemma 3.3, there is a unique extension $\alpha_p \in \check{\mathrm{H}}_d(\mathcal{M}_p)$, defined for all $p \in \mathcal{P}$, that satisfies (3.7). Write this extension α_p as $[\overline{\mathcal{M}}_p]^{vir}$. Then Axiom A1 holds, and we also have:

A2′. For every continuous path γ from p to q, the endpoint inclusions satisfy (3.4).

But A2′ is the special case of Axiom A2 with $Q = \gamma$ and $R = \partial\gamma$ and, in fact, is equivalent to Axiom A2, as follows. For any nonempty, compact, path connected subset Q of $\mathcal{P}$, define $[\overline{\mathcal{M}}_Q]^{vir} \in \check{\mathrm{H}}_d([\mathcal{M}_Q])$ to be $(\iota_p)_*\alpha_p$, where $\iota_p : p \hookrightarrow Q$ is the inclusion of some point in Q. This is well-defined, independent of p: given any $q \in Q$, we can find a path γ in Q between p and q, and then (3.7) shows that $(\iota_p)_*\alpha_p = (\iota_q)_*\alpha_q$ in $\check{\mathrm{H}}_d(\overline{\mathcal{M}}_\gamma)$, and therefore in $\check{\mathrm{H}}_d(\overline{\mathcal{M}}_Q)$. □

Theorem (3.5) has the following immediate consequence.

Corollary 3.6. *For any class $\alpha \in \check{\mathrm{H}}^*(\overline{\mathcal{M}})$, the function defined by*

$$I_p(\alpha) \;=\; \left\langle \alpha,\; [\overline{\mathcal{M}}_p]^{vir} \right\rangle \tag{3.12}$$

is constant in p on components of $\mathcal{P}$, and hence independent of p if $\mathcal{P}$ is connected. In particular, if $\mathcal{P}$ is connected, then for any continuous map $f : \overline{\mathcal{M}} \to Y$ there are invariants

$$I(\beta) \;=\; \left\langle f^*\beta,\; [\overline{\mathcal{M}}_p]^{vir} \right\rangle \;=\; \left\langle \beta,\; f_*[\overline{\mathcal{M}}_p]^{vir} \right\rangle$$

for each $\beta \in \check{\mathrm{H}}^*(Y)$.

We conclude this section with two finite-dimensional examples, both of which come from algebraic geometry. The first shows that the virtual fundamental class can be different from the actual fundamental class even when the fiber is a manifold.

Example 3.7 (Elliptic Surfaces)**.** An elliptic surface is a compact complex algebraic surface S with a holomorphic projection $\pi : X \to C$ to an algebraic curve C whose fiber is an elliptic curve except over a finite number of points $p_i \in C$. The singular fibers F_{p_i} are unions of rational curves, each possibly with singularities and multiplicities, and elliptic curves with multiplicity. The restriction of π to the union of the smooth fibers is a Fredholm map $X^* \to C$ of index 2, and $\pi : X \to C$ is a thin compactification of X^* regarded as a family over C.. Thus by Theorem 3.5, every fiber F_p carries a virtual fundamental class

$$[F_p]^{vir} \in \check{\mathrm{H}}_2(F_p, \mathbb{Z})$$

whose image $\iota_*[F_p]^{vir}$ in $\check{\mathrm{H}}_2(X, \mathbb{Q})$ is the homology class of the generic fiber.

In particular, if F_p is a smooth elliptic fiber with multiplicity $m > 1$, then F_p has a fundamental class $[F_p]$, but the virtual fundamental class is

$$[F_p]^{vir} = m[F_p]. \tag{3.13}$$

Example 3.8 (Lefschetz Pencils and Fibrations)**.** Consider a complex projective manifold X with a complete linear system $|D|$ of divisors of dimension at least 3. Lefschetz showed that a generic 2-dimensional linear system $[D]$ determines a holomorphic map $\pi : X \smallsetminus B \to \mathbb{P}^1$, where B is the base locus of $[D]$. The generic fiber of π is smooth and the other fibers have only quadratic singularities. This map π is Fredholm of index $d = \dim X - 2$. While π does not extend continuously to X, it does extend continuously over the blowup X_B of X along B, and $\tilde{\pi} : X_B \to \mathbb{P}^1$ is then a thin compactification of $X \smallsetminus B \to \mathbb{P}^1$. Theorem 3.5 therefore defines a virtual fundamental class

$$[F_p]^{vir} \in \check{\mathrm{H}}_d(F_p, \mathbb{Z})$$

on the fiber $F_p = \tilde{\pi}^{-1}(p)$ over each $p \in \mathbb{P}^1$.

4. Family expansions and thin coverings

In practice, applying Theorem 3.5 requires choosing a space $\mathcal{P}$ of parameters, and a compactification of $\mathcal{M} \to \mathcal{P}$. One then must stratify $S = \overline{\mathcal{M}} \smallsetminus \mathcal{M}$, and prove lemmas of two types:

(i) Formal dimension counts for all strata.

(ii) Transversality results showing that each stratum is a manifold of the expected dimension.

In general, (ii) can be done only if the space of parameters $\mathcal{P}$ is sufficiently large. Thus it may be necessary to enlarge the space of parameters in order to define virtual fundamental classes. Base expansions also are needed to show independence of added geometric structure, such as the choice of a Riemannian metric used to define Donaldson polynomials (see Section 5), and the choice of an almost complex structure used to define Gromov-Witten invariants (see Sections 6-10).

This section describes several ways to expand the base space $\mathcal{P}$ and extend virtual fundamental classes. The first type of expansion is to families where $\mathcal{M}$ may not be a Banach manifold over all of $\mathcal{P}$:

Lemma 4.1. *Let $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ be a proper continuous map from a Hausdorff space to a separable Banach manifold. Suppose that there is an open, dense and connected set $\mathcal{P}^*$ of $\mathcal{P}$ such that $\pi^{-1}(\mathcal{P}^*)$ is the thin compactification of a relatively oriented Banach manifold. Then $\pi : \overline{\mathcal{M}} \to \mathcal{P}$ admits a unique virtual fundamental class $[\overline{\mathcal{M}}_p]^{vir}$ for all $p \in \mathcal{P}$ (not just $p \in \mathcal{P}^*$).*

Proof. For each $p \in \mathcal{P}^*$, there is a virtual fundamental class $\alpha_p = [\overline{\mathcal{M}}_p]^{vir}$ by Theorem 3.5. The hypotheses on $\mathcal{P}^*$ imply that $p \mapsto \alpha_p$ satisfies the consistency condition (3.7). The proof of Theorem 3.5 then applies, without change, to show that a virtual fundamental class exists for all $p \in \mathcal{P}$. □

When enlarging the parameter space, some care is needed because the definitions of thin compactification and of virtual fundamental classes depend on the choice of $\mathcal{P}$. Thus enlarging the space of parameters may change the problem that one is trying to solve.

Definition 4.2. *A* base expansion *of the thin compactification* (3.2) *is a thin compactification of $\pi' : \mathcal{M}' \to \mathcal{P}'$ with a commutative diagram*

$$\begin{array}{ccc} \overline{\mathcal{M}} & \xrightarrow{\iota} & \overline{\mathcal{M}'} \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \overline{\pi}'} \\ \mathcal{P} & \hookrightarrow & \mathcal{P}' \end{array} \tag{4.1}$$

where (a) the bottom map is an inclusion as a submanifold, (b) the restriction of $\mathcal{M}'$ over $\mathcal{P}$ is $\mathcal{M}$, and (c) there is a second category subset $\mathcal{P}^*$ of $\mathcal{P}$ consists of regular values of both $\overline{\pi}$ and $\overline{\pi}'$ for each $p \in \mathcal{P}^*$.

Because the index is the dimension of a regular fiber, condition (c) implies that index π = index π'.

The following lemma shows that if a family already carries a VFC, expanding the space of parameters in the manner of Definition 4.2 does not change the VFC along the original family.

Lemma 4.3. *For every base expansion* (4.1)*, the virtual fundamental classes of π and π' agree over $\mathcal{P}$, i.e.*

$$\iota_*[\overline{\mathcal{M}}_p]^{vir} = [\overline{\mathcal{M}'}_p]^{vir} \tag{4.2}$$

in $\check{\mathrm{H}}_*(\overline{\mathcal{M}'}_p)$ for all $p \in \mathcal{P}$.

Proof. Conditions (a) and (b) of Definition 4.2 imply that $\mathcal{M} = \mathcal{M}'$ over $\mathcal{P}$, so for each p in the second category subset $\mathcal{P}^*$ of $\mathcal{P}$ appearing in Definition 4.2(c), $\mathcal{M}_p = \mathcal{M}'_p$ is a manifold with two thin compactifications $\overline{\mathcal{M}}_p$ and $\overline{\mathcal{M}'}_p$. Each carries a fundamental class by Theorem 2.3, and these are equal to the corresponding VFC by axiom A1. Therefore

$$\iota_*[\overline{\mathcal{M}}_p]^{vir} = \iota_*[\overline{\mathcal{M}}_p] = [\overline{\mathcal{M}'}_p] = [\overline{\mathcal{M}'}_p]^{vir} \qquad \forall p \in \mathcal{P}^*,$$

where the middle equality holds by Lemma 2.5, applied to the degree 1 map $\iota : \mathcal{M}_p \to \mathcal{M}'_p$. Then (4.2) follows by applying Extension Lemma 3.3, noting that the consistency condition (3.7) automatically holds for both $\iota_*[\overline{\mathcal{M}}_p]^{vir}$ and $[\overline{\mathcal{M}'}_p]^{vir}$ by Axiom A2′. □

Example 4.4. (a) If p is a regular point of $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$, then the inclusion of $\overline{\mathcal{M}}_p \to \{p\}$ into $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is a base expansion. Equation (4.2) becomes $[\overline{\mathcal{M}}_p] = [\overline{\mathcal{M}}_p]^{vir}$, which is Axiom A1 of Definition 3.4.

(b) Example 3.7 shows the importance of condition (c) in Definition 4.2. Let F_p be a smooth elliptic fiber in an elliptic surface with multiplicity $m > 1$. Then $F_p \to \{p\}$ is a thinly compactified family with $[F_p]^{vir} = [F_p]$, and the inclusion of $F_p \to \{p\}$ into $X \to C$ satisfies all of the conditions of Definition 4.2 except (c). But, as in (3.13), the virtual fundamental class induced by the enlarged family $X \to C$ is $m[F_p]$ rather than $[F_p]$.

(c) Similarly, in Example 2.4(a), the family $\pi_Z : \pi^{-1}(Z) \to Z$ embeds into $\pi : M_Z \to M$, but this is not a base expansion because no regular value for π_Z is regular for π. In this case, the dimensions of the generic fibers and the indices are different, and the two virtual fundamental classes are in different dimensions.

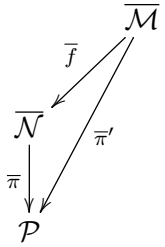
(d) For moduli spaces of solutions to an elliptic differential equation, a base expansion comes from lowering regularity of the parameters, for example, from $W^{k,p}$ to $W^{k-1,p}$. Often, elliptic theory implies that, for sufficiently large k and p, all conditions in Definition 4.2

are satisfied, and hence the virtual fundamental class is unchanged in the sense of Lemma 4.3.

Examples (b) and (c) above show that that the virtual fundamental class $[\overline{\mathcal{M}}]^{vir}$ *depends on the choice of the parameter space* $\mathcal{P}$. Thus it does not make sense to speak of "the" VFC of a single moduli space $\mathcal{M}_p$: virtual fundamental classes are, by their nature, associated with families over parameter spaces.

There is a parallel story for a second operation on thinly compactified families: lifting to covers.

Definition 4.5. *A* thin covering *of degree ℓ of a thinly compactified family $\overline{\mathcal{N}} \to \mathcal{P}$ is a thinly compactified family $\overline{\mathcal{M}} \to \mathcal{P}$ over the same parameter space with a commutative diagram of continuous maps*



(4.3)

such that

(a) there is a second category subset $\mathcal{P}^*$ of $\mathcal{P}$ consisting of regular values of both $\overline{\pi}$ and $\overline{\pi}'$, and

(b) for each $p \in \mathcal{P}^*$, every component $\mathcal{N}_{p,\alpha}$ of $\mathcal{N}_p$, contains a non-empty open set $U_{p,\alpha}$ so that the restriction of $f : \mathcal{M} \to \mathcal{N}$ to $f^{-1}(U_{p,\alpha})$ is a degree ℓ covering.

In Gromov-Witten theory, thin coverings arise by adding marked points and imposing constraints.

Lemma 4.6. *For every thin covering* (4.3) *of degree ℓ, the VFC of π and π' are related by*

$$\overline{f}_*[\overline{\mathcal{M}}_p]^{vir} \;=\; \ell\,[\overline{\mathcal{N}}_p]^{vir}. \tag{4.4}$$

Proof. At each $p \in \mathcal{P}^*$, the fibers $\overline{\mathcal{M}}_p$ and $\overline{\mathcal{N}}_p$ are thin compactifications of $\mathcal{M}_p$ and $\mathcal{N}_p$ respectively, and by Definition 4.5(b) and (2.4), their fundamental classes are related by $\overline{f}_*[\overline{\mathcal{M}}_p] \;=\; \ell\,[\overline{\mathcal{N}}_p]$. To see that this equality holds for all $p \in \mathcal{P}$, apply Extension Lemma 3.3, first with $\alpha_p = \overline{f}_*[\overline{\mathcal{M}}_p]^{vir}$, and then with $\alpha_p = \ell\,[\overline{\mathcal{N}}_p]^{vir}$, noting that (3.7) holds in both cases because virtual fundamental classes, by definition, satisfy Axiom A2′. □

The two operations described in this section can be used together. For example, in some situations a covering $\mathcal{N} \to \mathcal{M}$ can be specified by adding structure to the geometric objects representing points of $\mathcal{M}$. One can then expand the base $\mathcal{P}$ to a space $\mathcal{P}'$ of parameters that depend on this added structure.

5. Donaldson theory

Let X be a smooth, closed, oriented 4-manifold that satisfies the Betti number condition $b^+(X) > 1$. Donaldson theory uses moduli spaces of connections to construct invariants of the smooth structure of X. This section explains how Donaldson's polynomial invariants fit into the context of the previous sections. We follow Donaldson's exposition in Sections 5.6 and 6.3 of [D].

Fix a $U(2)$ vector bundle $E \to X$ with Chern number $k = (c_2(E) - \frac{1}{4}c_1^2(E))[X]$, and fix a connection ∇^0 on $\Lambda^2 E$. Let $\mathcal{A}_k(E)$ be the space of all connections on E that induce ∇^0 on $\Lambda^2 E$, and let $\mathcal{R}$ be the space of Riemannian metrics on X. After completing in Sobolev norms, the group $\mathcal{G}$ of gauge transformations of E with determinant 1 acts smoothly on $\mathcal{A}_k(E)$, and $\mathcal{B}_k = \mathcal{A}_k(E)/\mathcal{G}$ is an orbifold. Furthermore, both $\mathcal{R}$ and the subset $\mathcal{B}_k^* \subset \mathcal{B}_k$ of irreducible connections are Banach manifolds.

A pair (A, g) in $\mathcal{A}(E) \times \mathcal{R}$ is called an *instanton* if its curvature F^A satisfies $*ad(F^A) = -ad(F^A)$ where $*$ is the Hodge star operator on 2-forms for the metric g. The set of all $\mathcal{G}$-equivalence classes $([A], g)$ of instantons is the universal moduli space $\mathcal{M}_k \subset \mathcal{B}_k \times \mathcal{R}$. Projection onto the second factor is a map

$$\begin{array}{c} \mathcal{M}_k \\ \big\downarrow{\scriptstyle \pi_k} \\ \mathcal{R} \end{array} \tag{5.1}$$

whose restriction to $\mathcal{M}_k^* = \mathcal{M}_k \cap \mathcal{B}_k$ is a smooth Fredholm map of index $2d_k$, where

$$d_k = 4k - \tfrac{3}{2}(1 - b^1 + b^+). \tag{5.2}$$

Now assume that $c_1(E)$ is an odd element of $H^2(X;\mathbb{Z})/\text{Torsion}$. This implies that the space $\mathcal{A}_k(E)$ admits no flat connections [D, Section 5.6]. In fact, because $b^+(X) > 1$, there is a dense open and path connected subset $\mathcal{R}^*$ of $\mathcal{R}$ such that (5.1) is a manifold over $\mathcal{R}^*$ for all bundles E' with $c_1(E') = c_1(E)$ and Chern number $j \le k$ [DK, Section 4.3.3]. Hence by the Sard-Smale Theorem, there is a second category set of $g \in \mathcal{R}^*$ for which $\mathcal{M}_k(g) = \pi^{-1}(g)$ is a manifold of dimension $2d_k$. This is oriented by the choice of a homology orientation for X [DK, 7.1.39].

Lemma 5.1. *$\mathcal{M}_k \to \mathcal{R}$ extends to a proper map $\overline{\mathcal{M}}_k \to \mathcal{R}$ whose restriction to $\mathcal{R}^*$ is a thin compactification of $\mathcal{M}_k \to \mathcal{R}^*$.*

Proof. Using the topology of weak convergence [DK, Section 4.4], one sets

$$\overline{\mathcal{M}}_k \;=\; \mathcal{M}_k \cup \mathcal{S}, \tag{5.3}$$

where S is the union of the strata $S_{jk} = \mathcal{M}_{k-j} \times \text{Sym}^j(X)$ for $0 < j < k$ ($\mathcal{M}_0$ is empty because there are no flat connections). Then $\overline{\mathcal{M}}_k$ is paracompact, Hausdorff, and even metrizable [DK, Section 4.4]. Corollary A.2 in the appendix shows that $\mathcal{S}$ can be re-stratified to see that the family $\overline{\mathcal{M}} \to \mathcal{R}^*$ is a thin compactification. □

Theorem 3.5 produces a virtual fundamental class for the thin compactification $\overline{\mathcal{M}}_k \to \mathcal{R}^*$ of Lemma 5.1, and this extends over the entire space of metrics by Lemma 4.1. Thus we obtain a virtual fundamental class for Donaldson theory:

Corollary 5.2. *Let X be a closed, oriented 4-manifold with $b^+(X) > 1$, and let $E \to X$ a $U(2)$ vector bundle with Chern number k and $c_1(E)$ an odd element of $H^2(X;\mathbb{Z})/\mathit{Torsion}$. Then a homology orientation for X determines a virtual fundamental class $[\overline{\mathcal{M}}_k]^{vir}$ for the Uhlenbeck compactification*

$$\begin{array}{c} \overline{\mathcal{M}}_k \\ \big\downarrow{\scriptstyle \pi_k} \\ \mathcal{R}. \end{array} \tag{5.4}$$

To obtain invariants, one pairs $[\overline{\mathcal{M}}_k]^{vir}$ with the Čech cohomology classes defined by the μ-map

$$\mu : H_2(X;\mathbb{Q}) \to \check{\mathrm{H}}^2(\mathcal{B}^*_k;\mathbb{Q})$$

[DK, Chapter 5]. For each choice of classes $A_1, \dots, A_{d_k}$, the product $\mu(A_1) \cup \cdots \cup \mu(A_{d_k})$ restricts to a class

$$\mu(A_1, \dots, A_{d_k}) \in \check{\mathrm{H}}^{2d_k}(\mathcal{M}^*_k;\mathbb{Q}),$$

and under the inclusion $\iota_g : \mathcal{M}^*_k(g) \hookrightarrow \mathcal{M}^*_k$ of the moduli space over g further restricts to

$$\iota_g^* \mu(A_1, \dots, A_{d_k}) \in \check{\mathrm{H}}^{2d_k}(\mathcal{M}^*_k;\mathbb{Q}). \tag{5.5}$$

The next lemma shows that, for each metric g, the classes (5.5) are pullbacks under the inclusion $\iota_g : \mathcal{M}^*_k(g) \to \overline{\mathcal{M}}^*_k(g)$ into the compactification. The proof is dual to the proof of Lemma 3.3.

Lemma 5.3. *For each $A_1, \dots, A_{d_k} \in H_2(X;\mathbb{Z})$, there is a unique map*

$$g \mapsto \mu^g = \iota_g^* \mu(A_1, \dots, A_{d_k}) \in \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}^*_k(g);\mathbb{Q})$$

such that (i) μ^g is equal to the class (5.5) *for all regular $g \in \mathcal{R}^*$, and (ii) the consistency condition* (5.6) *below holds for all paths γ in $\mathcal{R}$.*

Proof. Donaldson and Kronheirmer showed [DK, Section 9.2.3] that for each regular g, μ^g has a Čech representative with compact support, so defines a class in the compactification

$$\mu^g \in \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}^*_k(g);\mathbb{Q}).$$

Furthermore, each pair $g, g' \in \mathcal{R}^*$ can be joined by a smooth path γ so that the compactified moduli space $\overline{\mathcal{M}}_k(\gamma)$ over γ contains no reducible connections. Then ι_g and $\iota_{g'}$ factor through the inclusion $\iota_\gamma : \overline{\mathcal{M}}_k(\gamma) \hookrightarrow \overline{\mathcal{M}}^*_k$, giving the consistency condition

$$\mu^g = \iota_g^* \mu^\gamma \text{ and } \mu^{g'} = \iota_{g'}^* \mu^\gamma \text{ for some } \mu^\gamma \in \check{\mathrm{H}}^{2d_k}(\overline{\mathcal{M}}^*_k(\gamma);\mathbb{Q}), \tag{5.6}$$

namely $\mu^\gamma = \iota_\gamma^* \mu(A_1, \dots, A_{d_k})$. Thus the hypotheses of Lemma 3.3 hold, with (3.6) replaced by $g \mapsto \mu^g$, and (3.7) replaced by the dual condition (5.6). The proof of Lemma 3.3 then applies in cohomology, with inverse limits replaced by direct limits, and the continuity property (1.12) used instead of (1.9). The lemma follows. □

Donaldson's polynomials are the maps $q_k : \mathrm{Sym}^{d_k} H_2(X;\mathbb{Q}) \to \mathbb{Q}$ defined as

$$q_k(A_1, \dots, A_{d_k}) \;=\; \left\langle \mu^g(A_1, \dots, A_{d_k}),\ [\overline{\mathcal{M}}_k(g)]^{vir} \right\rangle, \tag{5.7}$$

By Lemma 5.3(i), the righthand side agrees with Donaldson's original definition (which defined the invariants in terms of the moduli space over a regular g). Furthermore, the

righthand side is independent of the Riemannian metric $g \in \mathcal{R}$: given $A_1, \dots A_{d_k}$ and $g, g' \in \mathcal{R}$, choose a continuous path γ from g to g'. Then by Lemma 5.3(ii) the numbers

$$Q_g = \left\langle \mu^g,\ [\overline{\mathcal{M}}_k(g)]^{vir} \right\rangle$$

satisfy

$$Q_g - Q_{g'} \ = \ \left\langle \mu^\gamma,\ (\iota_g)_*[\overline{\mathcal{M}}_k(g)]^{vir} - (\iota_{g'})_*[\overline{\mathcal{M}}_k(g')]^{vir} \right\rangle,$$

which vanishes by Property A2′ in the proof of Theorem 3.5.

Thus the Donaldson polynomials (5.7) are invariants of the smooth structure of the manifold X, the class $c_1(E)$, and the homology orientation. In fact, changes in $c_1(E)$, and the homology orientation simply multiply the Donaldson polynomial by a (specific) sign. The story is completed by removing the assumption that E is admissible by using the stabilizing trick of Morgan and Mrowka; see [MM] or [D, Section 6.3].

This viewpoint makes clear that the invariance of Donaldson's polynomials follows directly from two core facts: (i) the Uhlenbeck compactification is a thin compactification over an open, dense, path connected subset $\mathcal{R}^*$ of the space of metrics, and (ii) $2d_k$-dimensional products of classes $\mu(A)$ extend to the compactification. Both appear explicitly in the work of Donaldson. As we have seen, these same two facts imply the existence of a virtual fundamental class $[\overline{\mathcal{M}}_k(g)]^{vir}$ for every metric g, and this encodes all needed cobordism arguments in Čech homology.

6. Gromov-Witten theory

In the remaining sections, we consider various thin compactifications in Gromov-Witten theory. This section summarizes the (well-known) setup; details are in [MS], [RT1] and [RT2].

The Deligne-Mumford spaces $\overline{\mathcal{M}}_{g,n}$ are at the foundation of Gromov-Witten theory. Points in $\overline{\mathcal{M}}_{g,n}$ represent equivalence classes $[C]$ of stable, nodal complex curves C of arithmetic genus g with n marked points $x_1, \dots x_n$; those without nodes form the principal stratum $\mathcal{M}_{g,n}$. There is a universal curve

$$\begin{array}{c} \overline{\mathcal{U}}_{g,n} \\ \big\downarrow \pi \\ \overline{\mathcal{M}}_{g,n} \end{array} \tag{6.1}$$

with the property that for each curve C as above there is a map $\varphi : C \to \overline{\mathcal{U}}_{g,n}$ whose image is a fiber of (6.1) that is biholomorphic (as a marked curve) to $C/\mathrm{Aut}(C)$. More generally, for any connected, n-marked genus g nodal curve C, there is a map $\psi : C \to \overline{\mathcal{U}}_{g,n}$ that first collapses all unstable irreducible components of C, and then applies φ.

Now let (X, ω) be a closed symplectic manifold and let $\mathcal{J} = \mathcal{J}(X)$ be the space of all ω-tame almost complex structures J on X. For each J, we consider maps $f : C \to X$ whose domain is an n-marked connected nodal curve C with complex structure j. Such a map is called J-holomorphic if

$$\overline{\partial}_J f = \tfrac{1}{2}(df + J df j) = 0, \tag{6.2}$$

and two such maps are regarded as equivalent if they differ by reparametrization. Let $\mathcal{M}_{A,g,n}(X)$ denote the moduli space of all equivalence classes of pairs (f, J), where $J \in \mathcal{J}$ and f is a J-holomorphic map with smooth stable domain that represents $A = [f(C)] \in H_2(X;\mathbb{Z})$. There is a projection and a stabilization-evaluation map

$$\begin{array}{ccc} \mathcal{M}_{A,g,n}(X) & \xrightarrow{se} & \overline{\mathcal{M}}_{g,n} \times X^n \\ \downarrow \pi & & \\ \mathcal{J} & & \end{array} \tag{6.3}$$

defined by $\pi(f, J) = J$ and $se(f, J) = ([C], f(x_1), \ldots, f(x_n))$.

More generally, each map $f : C \to X$ from a connected nodal curve has an associated graph map

$$\Gamma_f : C \to \overline{\mathcal{U}}_{g,n} \times X \tag{6.4}$$

defined by $\Gamma_f(x) = (\psi(C),\ f(x))$; this is an embedding if $\mathrm{Aut}(C) = 1$.Following Ruan and Tian [RT2, Definition 2.2], we can use Γ_f to expand the base of (6.3), as follows.

The universal curve $\overline{\mathcal{U}}_{g,n}$ is projective; fix an embedding $\overline{\mathcal{U}}_{g,n} \hookrightarrow \mathbb{P}^M$. For each fixed J, consider sections ν of the bundle $\mathrm{Hom}(\pi_1^* T\mathbb{P}^M, \pi_2^* TX)$ over $\mathbb{P}^M \times X$ that satisfy $J\circ\nu + \nu\circ j = 0$. Each such ν defines a deformation J_ν of the product almost complex structure on $\overline{\mathcal{U}}_{g,n} \times X$ by writing

$$J_\nu = \begin{pmatrix} j & 0 \\ -\nu \circ j & J \end{pmatrix}. \tag{6.5}$$

Definition 6.1. *Let $\mathcal{JV}(X)$ denote the space of smooth almost complex structures J_ν in the form (6.5); we write its elements as pairs (J, ν) and call them* Ruan-Tian perturbations.

A map $f : C \to X$ is (J, ν)-holomorphic if its graph satisfies $\overline{\partial}_{J_\nu} \Gamma_f = 0$, or equivalently if f satisfies

$$\overline{\partial}_J f(z) = \nu(z, f(z)). \tag{6.6}$$

Such a map is called *stable* if, for each irreducible component C_i of C, either C_i is stable or $f(C_i)$ is non-trivial in homology.

Now expand the base in (6.3) by the inclusion $\mathcal{J} \hookrightarrow \mathcal{JV}$ defined by $J \mapsto (J, 0)$. The maps π and se extend continuously over the universal moduli space $\overline{\mathcal{M}}_{A,g,n}(X)$ of all triples (f, J, ν) where f is a stable (J, ν)-holomorphic map, giving a diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{A,g,n}(X) & \xrightarrow{se} & \overline{\mathcal{M}}_{g,n} \times X^n \\ \downarrow \pi & & \\ \mathcal{JV}. & & \end{array} \tag{6.7}$$

We can now apply the following version of the Gromov Compactness Theorem, proved by Ivashkovich and Shevcheshin [IS, Theorem 1].

Theorem 6.2. *Fix $p > 2$. Every sequence $\{f_n : C_n \to X\}$ of J_n-holomorphic maps with fixed arithmetic genus and number of marked points, uniformly bounded energy, and with continuous J_n converging to J in C^0 has a subsequence that, after reparameterization, converges in C^0 and in $W^{1,p}$ to a stable J-holomorphic map $f : C \to X$.*

Replacing the maps f by their graphs Γ_f, one obtains the corresponding convergence statement for sequences of (J_n, ν_n)-holomorphic maps. This implies, in particular, that the projection π in (6.7) is a proper map.

The linearization of the (J,ν)-holomorphic map equation (6.2) at (f, J, ν) is an operator

$$\mathbf{D}_{f,J_\nu} : \Omega(f^*TX) \times H^{0,1}(C) \times T_J\mathcal{JV} \to \Omega^{0,1}(f^*TX)$$

given by formula [RT2, (3.10)]; see also [MS, Prop 3.1.1]. Restricting the last factor to be 0 gives the restricted linearization $D_f : \Omega(f^*TX) \times H^{0,1}(C) \to \Omega^{0,1}(f^*TX)$. After completing the spaces in (6.3) in appropriate Sobolev norms, one has two important regularity criteria (cf. [MS, Sections 3.1-3.2] and [RT2, Section 3]):

Reg 1. If $\mathbf{D}_{f,J_\nu}$ is surjective, the universal moduli space $\pi : \mathcal{M}_{A,g,n}(X) \to \mathcal{JV}$ in (6.3) is a manifold near (f, J, ν) with a natural relative orientation (see the proofs of [RT2, Theorem 3.2] or [MS, Theorem 3.1.5]).

Reg 2. If **Reg 1** holds, then at each regular value (J,ν) of π, the fiber $\mathcal{M}^{J,\nu}_{A,g,n}(X)$ is a manifold whose dimension is the index of D_f, which is

$$\iota(A,g,n) = 2[c_1(A) + (N-3)(1-g) + n] \tag{6.8}$$

where $\dim X = 2N$.

The construction of Gromov-Witten invariants now hinges on a single issue: *Find a thin compactification of* (6.3) *so that the map se extends over the compactification to give diagram* (6.7). Doing so, even over a portion of $\mathcal{JV}$, allows us to define the Gromov-Witten numbers

$$GW_{A,g,n}(\alpha) \;=\; \left\langle (se)^*\alpha,\; [\overline{\mathcal{M}}^J_{A,g,n}]^{vir} \right\rangle \qquad \text{for all } \alpha \in \check{\mathrm{H}}^*(\overline{\mathcal{M}}_{g,n} \times X^n; \mathbb{Q}). \tag{6.9}$$

Note that $\overline{\mathcal{M}}_{g,n} \times X^n$ is locally contractible, so by (1.13) α can also be regarded as an element of rational singular cohomology.

At this point, we can apply the results of Section 3, with the following payoffs:

(a) A thin compactification for the fiber over a single regular $J \in \mathcal{J}$ yields a virtual fundamental class $[\overline{\mathcal{M}}^J_{A,g,n}]^{vir}$. However, the numbers (6.9) may not be invariant under changes in J.
(b) A thin compactification over a connected neighborhood $\mathcal{P}$ of J_0 gives a virtual fundamental class at each $J \in \mathcal{P}$, and by Corollary 3.6 the numbers (6.9) are independent of J in $\mathcal{P}$.
(c) A thin compactification over all of $\mathcal{J}$ or $\mathcal{JV}$ gives numbers (6.9) that depend only on the symplectic structure of (X, ω).
(d) A thin compactification over the larger space $\mathcal{J}_{symp}$ of all tame pairs (ω, J), with ω varying, implies that the numbers (6.9) are invariants of the isotopy class of the symplectic structure on X.

We will take up the problem of constructing thin compactifications in the next section. Before proceeding, here are some simple examples that illustrate the ideas of this section.

Example 6.3 (Rational ghost maps)**.** For each $J \in \mathcal{J}(X)$, every J-holomorphic map $f : S^2 \to X$ representing the trivial class $A = 0$ is a constant map. It follows that D_f is the $\overline{\partial}$ operator on the trivial holomorphic bundle f^*TX, and f is regular because the sheaf cohomology group $H^1(S^2, f^*TX)$ vanishes. Hence for $n \geq 3$ the fibers of the moduli space $\overline{\mathcal{M}}^J_{0,0,n}(X) \to \mathcal{J}$ are all regular and canonically identified with $\overline{\mathcal{M}}_{0,n} \times X$. The virtual fundamental class $[\overline{\mathcal{M}}^J(X)]^{vir}$ is therefore equal to the actual fundamental class $[\overline{\mathcal{M}}_{0,n} \times X]$ and the GW invariants (6.9) are independent of J.

Example 6.4 (K3 surfaces)**.** Let X be a K3 surface, and consider the moduli space $\mathcal{M}(X) \to \mathcal{J}_{alg}$ of smooth rational holomorphic maps (f, J) for algebraic $J \in \mathcal{J}$. By a theorem of Mumford and Mori (see [MMu]), every algebraic K3 contains a non-trivial rational curve, so the fiber $\mathcal{M}^J_{A,0,0}(X)$ is non-empty for each algebraic J and some $A \neq 0$. But by (6.8) the index $\iota(A, 0, 0) = -2$ is negative. Thus $\mathcal{M}_{A,0,0}(X) \to \mathcal{J}_{alg}$ does not satisfy Reg 1. for any algebraic J.

Now extend the base by considering $\pi : \mathcal{M}(X) \to \mathcal{J}_{cx}$ over the space of all integrable complex structures. Each $J \in \mathcal{J}_{cx}$ determines a 20-dimensional subspace $H^{1,1}(X; \mathbb{R})$ of $H^2(X; \mathbb{R}) \cong \mathbb{R}^{22}$, and the resulting map $\mathcal{J} \to \mathrm{Gr}(20, 22)$ is a submersion. But $A \in H_2(X; \mathbb{Z})$ can be represented by a J-holomorphic curve only if the Poincaré dual of A is an integral $(1,1)$ class. It follows that $\mathcal{M}^J_{A,g,n}(X)$ is empty for all J in a subset $\mathcal{P} \subset \mathcal{J}_{cx}$ whose complement has codimension 2. Since empty fibers are regular, a virtual fundamental class exists over $\mathcal{P}$ and is equal to 0. This extends by Lemma 4.1, showing that

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} = 0$$

for all $A \neq 0$, g and n, and all $J \in \mathcal{J}_{cx}$, including the algebraic J.

Example 6.5 (Convex manifolds)**.** A complex algebraic manifold (X, ω, J) is called *convex* if $H^1(C, f^*TX) = 0$ for stable J-holomorphic maps $f : S^2 \to X$. Examples include projective spaces, Grassmannians, and Flag manifolds. Convexity implies that all J-holomorphic maps with smooth domain are regular, so $\mathcal{M}^J_{A,0,n}(X)$ is smooth and complex. It is also a quasi-projective variety (cf. [FP]), so its closure is a thin compactification. Hence there is a virtual fundamental class $[\overline{\mathcal{M}}^J_{A,0,n}(X)]^{vir}$ for the given J; more work is needed to determine if the associated GW numbers (6.9) are symplectic invariants.

7. Stable map compactification

The space of stable maps is the most commonly-used compactification of the moduli space (6.3) of smooth pseudo-holomorphic maps. Indeed, it is often regarded as the central object of Gromov-Witten theory. This section uses existing results to show that, in certain rather special circumstances, the space of stable maps is a thin compactification over $\mathcal{J}$. In these cases, the space of stable maps carries a virtual fundamental class.

Each stable map $f : C \to X$ has an associated dual graph $\tau(f)$, whose vertices correspond to the irreducible components C_i of C and whose edges corresponding to the nodes of C. Each vertex of the graph is labelled by the homology class $A_i = [f(C_i)] \in H_2(X; \mathbb{Z})$, by the

genus g_i of C_i, and by the number n_i of marked points on C_i. Every such graph τ defines a stratum $\mathcal{S}_\tau$ consisting of all stable maps f with $\tau(f) = \tau$. The space of all stable maps is the disjoint union

$$\overline{\mathcal{M}}_{A,g,n} \;=\; \mathcal{M}_{A,g,n} \,\cup\, \bigcup \mathcal{S}_\tau, \tag{7.1}$$

where $\mathcal{M}_{A,g,n}$ is as in (6.3), and where the union is over all non-trivial graphs τ with $\sum A_i = A$, $\sum n_i = n$, and with $\sum g_i$ plus the first Betti number of the graph equal to g. The Gromov Compactness Theorem implies that the projection $\overline{\pi} : \overline{\mathcal{M}}_{A,g,n} \to \mathcal{J}$ is proper.

To check whether (7.1) is a thin compactification one must, as always, compute the virtual dimension of each stratum, and prove transversality results that show that each stratum is a manifold over $\mathcal{J}$. In this case, the index calculations have been done many times in the literature (specifically, see Theorem 6.2.6(i) in [MS] or Section 4 in [RT1] for the $g = 0$ case, and Section 3 in [RT2] in general). These calculations show that, for each τ, the index of $\pi_\tau : \mathcal{S}_\tau \to \mathcal{J}$ is given by

$$\text{index } \pi_\tau \;=\; \iota(A,g,n) - 2k \tag{7.2}$$

where $\iota(A,g,n)$ is the index (6.8) of the principal stratum $\pi : \mathcal{M}_{A,g,n} \to \mathcal{J}$, and k is the number of nodes of the domain. Lemma A.1 in the appendix then shows that (7.1) is a thin compactification *provided all strata satisfy the transversality condition Reg. 1 in Section 6.*

Unfortunately, transversality can only be shown for certain classes of stable maps. One such class is:

Definition 7.1. *A stable map $f : C \to X$ is called* somewhere injective *if each irreducible component C_i of C contains a non-special point p_i such that $(df)_{p_i} \neq 0$ and $f^{-1}(f(p_i)) = \{p_i\}$.*

In the literature, it is usual to consider the universal moduli space of stable maps $\overline{\mathcal{M}} \to \mathcal{J}$ over the entire space of almost complex structures, and to show that the subset $\overline{\mathcal{M}}^*$ consisting of somewhere injective maps has good properties. Shifting perspective, we will restrict attention to the subset of $\mathcal{J}$ consisting of those J for which every map in $\overline{\mathcal{M}}^J$ is somewhere injective. Fixing (A,g,n), we set

$$\mathcal{J}_{s-inj} \;=\; \mathcal{J}_{s-inj}(A,g,n) \;=\; \Big\{ J \in \mathcal{J} \;\Big|\; \text{all } (f,J) \in \overline{\mathcal{M}}^J_{A,g,n} \text{ are somewhere injective} \Big\}.$$

Lemma 7.2. *For each (A,g,n), $\mathcal{J}_{s-inj}$ is an open subset of $\mathcal{J}$ in the C^0 topology.*

Proof. From Definition 7.1 and the discussion in [MS, Section 2.5], one sees that the complement of $\mathcal{J}_{s-inj}$ in $\mathcal{J}$ is the set of all J such that there exists a J-holomorphic map $f : C \to X$ in $\overline{\mathcal{M}}_{A,g,n}$ and an irreducible component C_i of C with either

(i) $f(C_i) = p$ is a single point,

(ii) the restriction $f|_{C_i}$ is a multiple cover of its image, or

(iii) there is another component C_j of C with $f(C_i) = f(C_j)$.

We will show that each of these is a closed condition on J, so the complement of $\mathcal{J}_{s-inj}$ is the union of three closed sets.

Suppose that a sequence $\{J_k\}$ converges in C^0 to $J \in \mathcal{J}$ and that there are stable J_k-holomorphic maps $f_k : C_k \to X$, and components $C_k' \subset C_k$ that satisfy (i). By Gromov compactness (6.2), after passing to a subsequences and then a diagonal subsequence, $\{f_k\}$ and $\{f_k|_{C_k'}\}$ converge in C^0 to J_0-holomorphic maps $f : C \to X$ and $f' : C_k' \to X$, respectively, for some nodal curves C and C'. But then $f' = f|_{C'}$ is a constant map, which means (i) is a closed condition on J.

If each $\{f_k|_{C'_k}\}$ is multiply covered then, by the proof of [MS, Proposition 2.5.1], there exit curves B_k and holomorphic maps $\varphi_k : C'_k \to B_k$ of degree > 1 such that $f_k|_{C'_k}$ is the composition $g_k \circ \varphi_k$ for a J_k-holomorphic map $g_k : B_k \to X$. Again by Gromov compactness, we may assume that, after restricting to C'_k, these converge to maps f', g and φ with $f' = g \circ \varphi$ and $\deg \varphi > 1$. Then $f' = f|_{C'}$ satisfies (ii), so (ii) is a closed condition on J.

The proof for (iii) is similar after using [MS, Corollary 2.5.3] to write $f_k|_{C_i}$ as the composition of $\varphi_k : C^i_k \to C^j_k$ and $g_k : C^j_k \to X$. □

Now fix (A, g, n) consider the restriction of the universal space of stable maps to $\mathcal{J}_{s-inj} = \mathcal{J}_{s-inj}(A, g, n)$ (assuming that it is not empty):

$$\begin{array}{c} \overline{\mathcal{M}}_{A,g,n} \\ \big\downarrow \pi \\ \mathcal{J}_{s-inj}. \end{array} \tag{7.3}$$

Lemma 7.3. *The family* (7.3) *is a thinly compactified family whose index* $d = \iota(A, g, n)$ *is given by* (6.8).

Proof. For each somewhere injective f, one can use the variation in the parameter $J \in \mathcal{J}$ to show that the linearization of the equation $\overline{\partial}_J f = 0$ (with fixed domain and map f) is onto. Specifically, for the $g = 0$ case, Proposition 6.2.7 and Theorem 6.3.1 in [MS] imply that each stratum $\mathcal{S}_\tau$ of (7.3) is a Banach manifold and $\pi_\tau : \mathcal{S}_\tau \to \mathcal{J}_{s-inj}$. has index given by (7.2). As mentioned before (6.8), the principal stratum is relatively oriented. Therefore (7.3) is a thinly compactified family when $g = 0$.

The same proofs (Propositions 6.2.7 and 6.2.8 and the proof of Theorem 6.3.1) in [MS] also apply for $g > 0$: they show that the linearization is surjective using variations that fix the complex structure on the domain, which implies, *a fortiori*, surjectivity as the domain is allowed to vary. □

Corollary 7.4. *The family* (7.3) *carries a virtual fundamental cycle*

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} \in \check{\mathrm{H}}_d(\overline{\mathcal{M}}^J_{A,g,n}(X); \mathbb{Q})$$

for all $J \in \mathcal{J}_{s-inj}(A, g, n)$. *The corresponding GW invariants* (6.9) *are constant on each path-component of* $\mathcal{J}_{s-inj}(A, g, n)$.

In particular, the Gromov-Witten invariants are invariant under C^0-small deformations of J in $\mathcal{J}_{s-inj}$. Note, however, that this is not enough to imply that they are symplectic invariants.

In a limited number of examples (X, A, g, n), one can show that $\mathcal{J}_{s-inj}$ is path connected and dense in $\mathcal{J}$. This occurs, for example, if energy considerations imply that all boundary strata are empty. Corollary 7.4 and Lemma 4.1 then define a virtual fundamental class over all of $\mathcal{J}$, and the corresponding GW invariants are symplectic invariants.

Example 7.5. For $X = \mathbb{C}P^N$, the universal space $\overline{\mathcal{M}}_{L,0,0}(X)$ of stable rational maps representing the class of a line is smooth and equal to $\mathcal{M}_{L,0,3}(X)$, and $\mathcal{J}_{s-inj}(L, 0, 0)$ is all of $\mathcal{J}$.

Example 7.6. Assume X is a Calabi-Yau 3-fold and $A \in H_2(X; \mathbb{Z})$ is a primitive homology class. Then there exists a subset $\mathcal{J}^*$ of $\mathcal{J}_{s-inj}(A, 0, 0)$ which is path connected, open and

dense in $\mathcal{J}$ (cf. [IP2, Lemmas 1.1 and 6.5]). In fact, the restriction to $\mathcal{J}^*$ of the universal space $\overline{\mathcal{M}}_{A,0,0}(X)$ of stable rational maps consists only of embeddings with smooth domains.

8. Domain-fine moduli spaces

As Examples 7.5 and 7.6 suggest, the somewhere injective condition is too restrictive for most applications. In the genus 0 case, the needed transversality results hold for the slightly larger class of maps ("simple maps") that are somewhere injective on the complement of ghost components; see [MS, Example 6.2.5]. But it is more effective to expand the base space $\mathcal{J}$ to the space $\mathcal{JV}$ of Ruan-Tian perturbations.

Definition 8.1. *A (J,ν)-holomorphic map $f: C \to X$ is called* domain-fine *if* $\operatorname{Aut} C = 1$.

Any domain-fine map $f : C \to X$ is stable. Furthermore, the map $C \mapsto \psi(C)$ into the universal curve (6.1) is an embedding, and hence the graph map (6.4) is somewhere injective. While the proofs in the previous section do not immediately apply (because of the 0 in (6.1)), their conclusions holds, as we show next.

Fix (A,g,n) and set

$$\mathcal{J}_{d-f} \;=\; \mathcal{J}_{d-f}(A,g,n) \;=\; \Big\{ J \in \mathcal{J} \;\Big|\; \text{all } (f,J) \in \overline{\mathcal{M}}^J_{A,g,n} \text{ are domain-fine} \Big\}.$$

As in Lemma 7.2, Gromov compactness implies that $\mathcal{J}_{d-f}$ is an open subset of $\mathcal{J}$.

Lemma 8.2. *$\mathcal{J}_{d-f} = \mathcal{J}_{d-f}((A,g,n)$ is an open subset of $\mathcal{J}$ in the C^0 topology. The space $\mathcal{JV}^\ell_{d-f}$, defined similarly, is also open.*

Proof. Under Gromov convergence, the order of the automorphism group is upper semi-continuous, and limits of unstable domain components are unstable. Thus each domain-fine map f has a neighborhood with the same property. For $J \in \mathcal{J}_{d-f}$, these open sets cover the moduli space $\overline{\mathcal{M}}^J_{A,g,n}(X)$, and hence by compactness cover the moduli spaces $\pi^{-1}(U)$ for some open neighborhood U of J. The same argument applies to $\mathcal{JV}^\ell_{d-f}$. □

Example 8.3. Because stable rational curves have no non-trivial automorphisms, the space $\overline{\mathcal{M}}_{0,0,n}(X)$ of rational ghost maps considered in Example 6.3 is domain-fine for all J, so in this case $\mathcal{J}_{d-f}(0,0,n) = \mathcal{J}$.

The next proposition rephrases the main result of Ruan and Tian in [RT1] and [RT2]. It shows that the techniques developed in Sections 1-3 apply directly to domain-fine moduli spaces.

Proposition 8.4. *Fix (A,g,n) and $\mathcal{JV}_{d-f}$ as in Lemma 8.2. Then the restriction of the universal moduli space of stable maps $\overline{\mathcal{M}}_{A,g,n}(X) \to \mathcal{JV}^\ell_{d-f}$ is a thinly compactified family of index $d = \iota(A,g,n)$.*

Proof. For domain-fine maps, one can use the variation in ν to show that the linearization of the equation $\overline{\partial}_J f = \nu$ is onto (essentially because the graph Γ_f of f is an embedding, thus somewhere injective). The proof is completed exactly as in the proof of Lemma 7.3. □

Corollary 8.5. *For each (A, g, n), the thin compactification of Proposition 8.4 determines a unique virtual fundamental class*

$$[\overline{\mathcal{M}}^J_{A,g,n}(X)]^{vir} \in \check{\mathrm{H}}_*(\overline{\mathcal{M}}^J_{A,g,n}(X), \mathbb{Q}) \tag{8.1}$$

defined for $J \in \mathcal{JV}_{d-f}$. The corresponding GW numbers (6.9) *are independent of J on each path component of $\mathcal{J}_{d-f}$.*

Proof. Apply Theorem 3.5 to the family of Lemma 8.4. □

A priori, $\mathcal{J}_{d-f}$ may be empty or have many components. Thus Corollary 8.5 is not yet enough to define a virtual fundamental class over $\mathcal{JV}$. In particular, for $g = 0$, the stable map compactification $\overline{\mathcal{M}}_{A,0,n}(X) \to \mathcal{JV}$ may not be a thin family over the entire $\mathcal{JV}$ because of the presence of multiply-covered unstable domain components. On such components, the perturbation ν vanishes and cannot be used to verify condition **Reg 1** of Section 6.

Compactifications that are not thin occur in Gromov-Witten theory, as in the following example.

Example 8.6. Fix a complex structure j on a smooth torus T with one marked point, and let $\mathcal{M}$ denote the moduli space of degree d stable maps $f : T \to \mathbb{P}^2$ for fixed j. Then $\mathcal{M}$ is a smooth family over the space $\mathcal{J}$ of almost complex structures on $\mathbb{P}^2$, and this family has a bubble tree compactification $\overline{\mathcal{M}}$. However, the results of [I1] show that the restriction $\overline{\mathcal{M}} \to \mathcal{J}$ is not a thin compactification: it contains a stratum (maps whose domain is a ghost torus with a degree d rational bubble tree) which is always larger dimensional than $\mathcal{M}$.

This difficulty can be resolved in two ways:

(i) $\mathcal{M}$ and $\overline{\mathcal{M}}$ extend to a thinly compactified family over $\mathcal{JV}$ (using [RT1]).

(ii) There is a different, smaller, thin compactification of $\mathcal{M} \to \mathcal{J}$, which includes maps f from nodal rational curves with a ghost torus attached at a point p only if $df(p) = 0$; see [I1].

In this case, Lemma 4.3 does not apply, and the two virtual fundamental classes are different. The difference between the corresponding invariants is calculated in [I1].

9. Domain-fine relative moduli spaces

The virtual fundamental classes constructed in Section 8 come with a big caveat: they exist only if $\mathcal{J}_{d-f}$ is non-empty, that is, if there is some J for which the domain C of every stable J-holomorphic map is stable with $Aut(C) = 1$. In general, there is no such J. A natural way to proceed is to add some geometric structure that has the effect of introducing more special points on all domains, and hope that these new points stabilize and rigidify all domains. This procedure may yield moduli spaces with several different compactifications, and one must ask if any of these are thin. Below, we do this for added structures of two types: a smooth symplectic divisor, and a normal crossings divisor. In each case we show that existing results in the literature are enough to construct a virtual fundamental class for domain-fine J.

An embedded codimension 2 submanifold V of (X,ω) will be called a *divisor* if it is J-holomorphic for some ω-tame J. A finite union of divisors in general position is a *normal crossing divisor*, also denoted by V, if it is J-holomorphic for some ω-tame J (see [I2, Definition 1.3] for precise definition). In either case, there are associated subspaces $\mathcal{J}^V \subset \mathcal{J}$ and $\mathcal{JV}^V \subset \mathcal{JV}$ consisting of all J (respectively (J,ν)) that satisfy a condition ("V-adapted") on the 1-jet of J along V as defined in Definition 3.2 of [IP1] and Section A.2 of [I2]. We assume that these subspaces are non-empty; this is true for any divisor V (see [IP1]), but is an assumption on V for normal crossing divisors.

In [IP1] we defined relative moduli spaces $\mathcal{M}_s(X,V)$ consisting of stable maps with smooth domain whose image intersects V at points with multiplicities $s=(s_1,\ldots,s_\ell)$. We also constructed a relative stable map compactification

$$\pi : \overline{\mathcal{M}}_s(X,V) \to \mathcal{JV}^V \tag{9.1}$$

that consists of equivalence classes of certain types of maps $f : C \to X_m$ into a "level m building" $X_m = X \cup \mathbb{P}_V \cup \cdots \cup \mathbb{P}_V$, modulo the action of $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ where $\mathbb{C}^*$ acts fiberwise on the projectivized normal bundle $\mathbb{P}_V = \mathbb{P}(N_V \oplus \mathbb{C})$. There is a refined se map which also keeps track of the multiplicities s. This was extended in [I2] to the case V is a normal crossing divisor; s then records multiplicities of intersections with each statum of V, and the evaluation map now also keeps track of the leading coefficients of f at the $\ell(s)$ contact points to V (as elements of a weighed projectivization $\mathbb{P}_s(NV)$ of the normal bundle to V, cf. [I2, (3.17)]). In both cases, for each (A,g,n,s), the maps π and se are continuous on the components $\overline{\mathcal{M}}_{A,g,n,s}(X,V)$ of (9.1), giving a diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{A,g,n,s}(X,V) & \xrightarrow{se} & \overline{\mathcal{M}}_{g,n+\ell(s)} \times X^n \times \mathbb{P}_s(NV) \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{JV}^V & & \end{array} \tag{9.2}$$

There is also a continuous forgetful map

$$\varphi_V : \overline{\mathcal{M}}_{A,g,n,s}(X,V) \to \overline{\mathcal{M}}_{A,g,n+\ell(s)}(X) \tag{9.3}$$

induced by composing $f : C \to X_m$ with the collapsing map $X_m \to X$, and then contracting all unstable domain components whose image is a single point. The image of a $f \in \overline{\mathcal{M}}_s(X,V)$ is a map $f_0 : C_0 \to X$ that may have components in V:

$$\begin{array}{ccc} C & \xrightarrow{f} & X_m \\ \downarrow{\scriptstyle \mathrm{ct}} & & \downarrow{\scriptstyle \pi_X} \\ C_0 & \xrightarrow{f_0} & X \end{array} \tag{9.4}$$

Note that the contracted domain $C_0 = \mathrm{ct}(C)$ of f depends on f (not just C), and is different from the stable model $\mathrm{st}(C)$, which is obtained by contracting all unstable domain components, not just the trivial ones.

Definition 9.1. *A map $f \in \overline{\mathcal{M}}(X,V)$ is called* domain-fine *if $f_0 \in \overline{\mathcal{M}}(X)$ is domain-fine, that is, if the contracted domain C_0 is stable with $Aut(C_0) = 1$.*

We now proceed as in previous sections. Again, the subset of the moduli space (9.2) consisting of domain-fine maps is a manifold, and we restrict attention to those fibers of π

that are entirely contained in this subset. Thus we fix (A, g, n, s) and set

$$\mathcal{JV}^V_{d-f} \;=\; \Big\{(J,\nu) \in \mathcal{JV}^V \;\Big|\; \text{all } (f,J,\nu) \in \overline{\mathcal{M}}^{J,\nu}_{A,g,n,s} \text{ are domain-fine}\Big\};$$

this is an open subset of $\mathcal{JV}^V$ that implicitly depends on (A, g, n, s). Each domain-fine map has irreducible components $f_i : C_i \to X$ of two types: (i) ones whose graph map (6.4) is somewhere injective, and (ii) trivial components, i.e. unstable domain components that multiply-cover a fiber of one $\mathbb{P}_V$ in X_m. On trivial components the perturbation ν vanishes identically and the restricted linearization D_{f_i} is the $\overline{\partial}$-operator on the holomorphic bundle $TC_i \oplus E$, where E is holomorphically trivial. Hence, coker $D_{f_i} = 0$ and, as in Example 6.3, this allows one to obtain all needed transversality results at domain-fine maps, as is shown in Lemma 5.23 in [I2].

Proposition 9.2. *Over $\mathcal{JV}^V_{d-f}$, the universal moduli space of relatively stable maps*

$$\overline{\mathcal{M}}_{A,g,n,s}(X,V) \xrightarrow{\pi} \mathcal{JV}^V_{d-f} \tag{9.5}$$

is a thinly compactified family. Therefore for each $J \in \mathcal{J}^V_{d-f}$, there is a virtual fundamental class

$$[\overline{\mathcal{M}}^J_{A,g,n,s}(X,V)]^{vir}.$$

Its pushforward under the se map in (9.2) *is the relative GW invariant over $\mathcal{JV}^V_{d-f}$ defined in [IP1, Theorem 8.1] and [I2, Theorem 8.1].*

Proof. If V is a smooth divisor, Theorem 7.4 of [IP1] shows that (9.5) is a thinly compactified family (see also Lemmas 7.5 and 7.6, and note that domain-fine maps f are "irreducible" in the sense of [IP1, Definition 1.7] and are smooth points of their stratum of the universal moduli space because $\mathrm{Aut}(C_0) = 1$). The corresponding facts for normal crossing divisors are proved in [I2]: π is proper by Theorems 7.5, and by Theorem 7.6 every stratum of (9.5) is smooth, and all boundary strata have index at least 2 less than the index of the principal stratum, so again (9.5) is a thinly compactified family. One can then apply Theorem 3.7 and Corollary 3.6. □

Small relative compactification When V is smooth, there is a smaller compactified relative moduli space

$$\overline{\mathcal{N}}(X,V)$$

that is sometimes easier to work with, and was recently used by J. Pardon; see Remark 1.9 in [Pd2]. Each point in $\overline{\mathcal{M}}(X,V)$ is represented by a map $f : C \to X_m$ whose restrictions $f_i = f|_{C_i}$ to the irreducible components C_i of C are of two types: "level 0" maps $f_i : C_i \to X$ with images not in V, and "level k" components whose image lies in the k^{th} copy of $\mathbb{P}_V$, $k \geq 1$. Some positive level components may be trivial, but each positive level has at least one nontrivial component. Then $\overline{\mathcal{N}}(X,V)$ is obtained from this set of maps by contracting all unstable domain components whose image is a fiber of $\mathbb{P}_V$ and taking the quotient by the action of $\mathbb{C}^*$ *on each nontrivial component.* This is a quotient of $\overline{\mathcal{M}}(X,V)$, and the map (9.3) factors through this compactification:

$$\overline{\mathcal{M}}_{A,g,n,s}(X,V) \xrightarrow{\psi} \overline{\mathcal{N}}_{A,g,n,s}(X,V) \xrightarrow{\varphi'_V} \overline{\mathcal{M}}_{A,g,n+\ell(s)}(X) \tag{9.6}$$

where φ'_V is the composition with the collapsing map $X_m \to X$. As in Proposition 9.5, $\overline{\mathcal{N}}_{A,g,n,s}(X,V)$ is also a thin compactification of the moduli space $\mathcal{M}(X,V)$ over $\mathcal{JV}^V_{d-f}$. The labelling of the boundary strata is the same, but their dimension may decrease: a

relatively stable map $f : C \to X_m$ in $\overline{\mathcal{M}}(X,V)$ lies in a stratum of codimension equal to twice the maximun level of the non-trivial components of f, while the corresponding stratum of $\overline{\mathcal{N}}(X,V)$ is twice the total number of nontrivial components with positive level.

Because these are two thin compactifications of the same space $\mathcal{M}_{A,g,n,s}(X,V)$, Lemma 2.5 applies with $\ell = 1$ to show that

$$\psi_*[\overline{\mathcal{M}}_{A,g,n,s}(X,V)]^{vir} \;=\; [\overline{\mathcal{N}}_{A,g,n,s}(X,V)]^{vir}$$

over $\mathcal{JV}^V_{d-f}(A,g,n,s)$.

10. Applications: the Semipositive and Genus 0 cases

We close by mentioning – without details – two important cases in which Gromov-Witten invariants have been defined, and where the existing proofs can be reinterpreteted as showing the existence thin compactifications.

Semipositive manifolds. In general, spaces of stable maps $\overline{\mathcal{M}}_{A,g,n}(X) \to \mathcal{JV}$ are not thinly compactified families because of the presence of multiply-covered unstable domain components with negative Chern class. But Ruan and Tian observed that if X is semipositive (cf. Definition 6.4.1 in [MS] or page 456 of [RT2]), there is a quotient

$$\overline{\mathcal{N}}_{A,g,n}(X) \to \mathcal{JV} \tag{10.1}$$

of the space of stable maps whose boundary strata all have codimension at least 2 for generic (J,ν). This quotient, which they call the *GU-compactification*, is obtained by replacing multiply-covered unstable components by their images; the basic idea is described in Section 3 of [RT2], and a detailed description in the $g = 0$ case is given in Sections 6.1 and 6.4 of [MS]. The transversality lemmas and dimension counts in these references prove most of what is needed to show that (10.1) is a thinly compactified family (cf. Theorem 3.11 of [RT2]).

Granting this, Theorem 3.5 implies the existence of a virtual fundamental class, as an element in the rational Čech homology of $\overline{\mathcal{N}}^J_{A,g,n}(X)$. Since the stabilization-evaluation map *se* factors through the GU-compactification, this defines GW invariants as in (6.9) for all semipositive closed symplectic manifolds.

Remark 10.1. In fact, Ruan and Tian assert that if $\overline{\mathcal{N}}^J_{A,g,n}(X)$ is an analytic space, then it carries a virtual fundamental class in singular homology (Remark 3.12 in [RT2]). It may be possible to prove this. But it is more natural to work with rational Čech homology as above, in which case the existence of a virtual fundamental class follows from the results in [RT2].

Genus 0 GW invariants. Suppose that X is a closed symplectic manifold whose symplectic form ω represents a rational class in $H^2(X;\mathbb{R})$. Then Donaldson's Theorem shows that for each almost complex structure J_0 and each $k \gg 0$ there is a submanifold V_k that is Poincaré dual to $[k\omega]$. Let $\mathcal{J}_0(V_k)$ be the space of all J such that V_k is a J-holomorphic.

Then for $k \gg 0$ and $g = 0$, there exists an open, dense and path connected subset $\mathcal{J}^*$ of $\mathcal{J}_0(V_k)$ over which the relative stable map compactifications

$$\overline{\mathcal{M}}_{A,0,n,s}(X, V_k) \to \mathcal{J}^* \tag{10.2}$$

are domain-fine, and the only maps in V_k are constant maps (cf. [CM, Corollary 8.16]).

K. Cieliebak and K. Mohnke exploit this fact to define genus 0 Gromov-Witten invariants on X. They define spaces $\mathcal{J}_{l+1}(V_k, \dots)$ of domain-dependent almost complex structures and show that the evaluation map factors through a quotient $\overline{\mathcal{N}}_{A,0,n,s}(X, V_k)$ of (10.2):

$$\overline{\mathcal{M}}_{A,0,n,s}(X, V_k) \longrightarrow \overline{\mathcal{N}}_{A,0,n,s}(X, V_k) \xrightarrow{ev} X^n. \tag{10.3}$$

Transversality lemmas and dimension counts show that the boundary strata of $\overline{\mathcal{N}}$ all have codimension at least 2 for generic J in $\mathcal{J}_{l+1}(V_k, \dots)$ (cf. [CM, Proposition 9.6]). These proofs, reinterpreted, show that for $s = (\mathbf{1}) = (1, \dots, 1)$, $\overline{\mathcal{N}}_{A,0,n,(\mathbf{1})}(X, V_k)$ is a thin compactification of $\mathcal{M}_{A,0,n,(\mathbf{1})}(X, V_k)$ over an open dense subset of $\mathcal{J}_{l+1}(V_k, \dots)$. Theorem 3.5 yields a virtual fundamental class for this family, which by Lemma 4.1 extends to a class in rational Čech homology for the family over $\mathcal{J}_0(V_k)$. Using the evaluation map ev in (10.3), one then has V_k-dependent Gromov-Witten numbers

$$\begin{aligned} GW_{A,g,n}(\alpha) &= \frac{1}{(A \cdot V_k)!} \left\langle ev^*\alpha,\ [\overline{\mathcal{N}}_{A,0,n,(\mathbf{1})}(X, V_k)]^{vir} \right\rangle \\ &= \frac{1}{(A \cdot V_k)!} \left\langle \alpha,\ ev_*[\overline{\mathcal{N}}_{A,0,n,(\mathbf{1})}(X, V_k)]^{vir} \right\rangle \end{aligned}$$

for all $\alpha \in H^*_{sing}(X^n; \mathbb{Q})$.

Finally, Cieliback and Mohnke show that these numbers are independent of the choice of the submanifold V_k (see [CM, Section 10]), and hence are symplectic invariants. It is conjectured that these agree with the standard GW invariants whenever both are defined.

Appendix

In practice, compactifications of moduli spaces often come with natural stratifications. The following lemma provides a useful criterion that implies that a compactification is thin.

Lemma A.1. *Consider an index ι family* (2.1) *such that there exists a Hausdorff space $\overline{\mathcal{M}}$ containing $\mathcal{M}$ as an open set and an extension of π to a proper continuous map $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ such that*

(a) $\overline{\mathcal{M}}$ can be written as a disjoint union of sets $\{\mathcal{S}_\alpha | \alpha \in \mathcal{A}\}$ indexed by a finite set $\mathcal{A}$ with $\mathcal{S}_0 = \mathcal{M}$.
(b) Each $\mathcal{S}_\alpha$ is a manifold, and $\pi_\alpha = \overline{\pi}|_{\mathcal{S}_\alpha}$ is a smooth Fredholm map $\mathcal{S}_\alpha \to \mathcal{P}$ of index ι_α.
(c) $\iota_\alpha \le \iota - 2$ for all $\alpha \neq 0$, and

$$\overline{\mathcal{S}}_\alpha \setminus \mathcal{S}_\alpha \subseteq \bigcup_{\{\beta \,|\, \iota_\beta < \iota_\alpha\}} \mathcal{S}_\beta.$$

Then $\overline{\pi} : \overline{\mathcal{M}} \to \mathcal{P}$ is a thinly compactified family.

Proof. Condition (c) implies that the accumulation points of $\mathcal{S}_\alpha$ must lie in strata of strictly smaller index. Hence for each k, the union of strata of index $\iota - k$

$$X_k = \bigcup_{\iota_\alpha = \iota - k} \mathcal{S}_\alpha$$

is topologically a disjoint union of manifolds. This means that each X_k is a manifold, and that the restriction of $\overline{\pi}$ to X_k is a Fredholm map of index $\iota - k$. We can then apply Lemma 2.2 to conclude that

$$\overline{\mathcal{M}} \;=\; \mathcal{M} \cup \bigcup_{\alpha \neq 0} \mathcal{S}_\alpha \;=\; \mathcal{M} \cup \bigcup_{k \geq 2} X_k$$

is a thin compactification of $\mathcal{M} \to \mathcal{P}$. □

As one application, consider the Uhlenbeck compactification (5.3). We will show that $\overline{\mathcal{M}}_k$ has a stratification – different from the one in (5.3) – that satisfies the hypotheses of Lemma A.1. The strata are labeled by partitions.

A partition is a non-increasing sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers; its length $\ell(\alpha) = \ell$ and its weight $|\alpha| = \sum \alpha_i$ satisfy $\ell(\alpha) \leq |\alpha|$. We also consider (0) to be a partition with $\ell(0) = |(0)| = 0$. Let $\mathcal{P}_k$ be the set of all partitions α with $|\alpha| \leq k$. Define the *level* of α to be

$$\Lambda(\alpha) = 2|\alpha| - \ell(\alpha), \tag{A.1}$$

and note that $\Lambda(\alpha) \geq 0$ with equality if and only if $\alpha = (0)$.

Given a four-manifold X, an integer $k \geq 0$, regard $\mathrm{Sym}^k X$ as formal positive sums $\sum \alpha_i x_i$ of distinct points of X associated with some partition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $|\alpha| = k$. Let Δ_α be the set of all such sums associated with a given α. Then Δ_α is a manifold of dimension $4\ell(\alpha)$, and $\mathrm{Sym}^k X$ is the disjoint union of the sets Δ_α over all α with $|\alpha| = k$. With these preliminaries understood, we can prove a fact used in the proof of Lemma 5.1.

Corollary A.2. *With $\mathcal{R}^*$ as Section 5, the Uhlenbeck compactification $\overline{\mathcal{M}}_k \to \mathcal{R}^*$ is a thin compactification.*

Proof. Re-stratify the compactification (5.3) by writing

$$\overline{\mathcal{M}}_k \;=\; \mathcal{M}_k \;\cup \bigcup_{\alpha \in \mathcal{P}_k} \mathcal{S}_\alpha$$

where

$$\mathcal{S}_\alpha \;=\; \mathcal{M}_{k-|\alpha|} \times \Delta_\alpha.$$

By the choice of $\mathcal{R}^*$, each $\mathcal{S}_\alpha$ is a Banach manifold with a Fredholm projection $\pi_\alpha : \mathcal{S}_\alpha \to \mathcal{R}^*$ of index

$$\iota_\alpha \;=\; 2d(k - |\alpha|) + 4\ell(\alpha) \;=\; 2d_k - 4\Lambda(\alpha), \tag{A.2}$$

where d_k is the dimension of the top stratum $\mathcal{M}_k$.

One then sees that conditions (a) and (b) of Lemma A.1 hold. To verify (c), suppose that a sequence $(A_n, \sum \alpha_i(x_n)_i)$ converges in the weak topology. Then $\{A_n\}$ converges to a formal instanton $(B, \sum \beta_j y_j)$ with $B \in \mathcal{M}_{k-|\alpha|-|\beta|}$, and $\sum \alpha_i(x_n)_i$ converges to $\sum \gamma_m z_m$ with $\ell(\gamma) \leq \ell(\alpha)$ and $|\gamma| = |\alpha|$. Thus the limit is

$$\Big(B, \sum \beta_j y_j + \sum \gamma_m z_m\Big) \in \mathcal{M}_{k-|\delta|} \times \Delta_\delta,$$

with $\ell(\delta) \leq \ell(\beta) + \ell(\gamma) \leq \ell(\alpha) + \ell(\beta)$ and $|\delta| = |\beta| + |\gamma| = |\alpha| + |\beta|$. The level (A.1) of this limit stratum is therefore

$$\Lambda(\delta) \;=\; 2|\delta| - \ell(\delta) \;\geq\; \Lambda(\alpha) + \Lambda(\beta) \geq \Lambda(\alpha),$$

with equality if and only if $\beta = (0)$ and $\gamma = \alpha$. This, together with (A.2), implies property (c) of Lemma A.1. The corollary follows. □

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