Topics in Gauge Theory

Problem Set 5

Due Weds, Nov 27

Exercise 11. Let $\gamma(t)$ be a path in a Riemannian manifold (M, g) with tangent vector $T = \dot{\gamma}$. Define a Lagrangian $L : TM \to \mathbb{R}$ by $L = KE - U = KE = \frac{1}{2}|T|^2$. In local coordinates, $\gamma(t) = (x^1(t), \cdots, x^n(t))$ and

$$L(x^i, \dot{x}^i) \;=\; \tfrac{1}{2} \sum g_{ij}\, \dot{x}^i \dot{x}^j$$

Show that γ satisfies the Euler-Lagrange equations for L if and only if it is a solution of the geodesic equation

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = 0$$

where Γ^i_{jk} are the Christoffel symbols, defined by

$$\Gamma^i_{jk} \;=\; \frac{1}{2} \sum g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right).$$

Exerecise 12. Suppose that (M, ω) is a symplectic manifold with Hamiltonian function H, and that $\{x^i, p^i\}$ is a local Darboux coordinate chart.

(a) Use the formula for the Poisson bracket in terms of partial derivatives to show that

$$\{x^i, x^j\} = \{p^i, p^j\} = 0 \qquad \text{and} \qquad \{x^i, p^j = \delta^{ij}.$$

(b) Recall that observables $f \in C^\infty(M)$ evolve by $\dot{f} = \{f, H\}$. Show that this formula reduces to Hamilton's equations when $f = x^i$ and p^i.

Exerecise 13. Show that the Schrodinger equation $i\hbar \partial_t \psi = H\psi$ on a complex Hilbert space V induces a flow on the projective space $\mathbb{P}(V) = V - \{0\}/\mathbb{C}^*$ whose fixed points are eigenvectors ψ of H.