Topics in Gauge Theory

Problem Set 3

Due Friday, Oct 25

Exercise 8. Let $\mathcal{G} = Map(M, S^1)$ be the gauge group of a principal S^1-bundle $P \to M$. The component $\mathcal{G}_1$ of $\mathcal{G}$ containing the identity has a normal subgroup isomorphic to S^1 consisting of the maps from M to S^1 whose image is a single point, and there is an an exact sequence

$$1 \to S^1 \to \mathcal{G}_1 \to \mathcal{G}_0 \to 1$$

where $\mathcal{G}_0$ is the quotient group $\mathcal{G}_1/S^1$. Show that $\mathcal{G}_0$ is isomorphic to the subgroup

$$\mathcal{G}_2 \;=\; \left\{ g = e^{if} \in \mathcal{G}_1 \;\middle|\; \int_M f = 0 \;\right\}.$$

Exercise 9. In class we proved that for a first-order elliptic operator $D : \Gamma(E) \to \Gamma(F)$ on a compact Riemannian manifold, there is a L^2-orthogonal decomposition

$$L^2(V) = ker\ D \,\oplus\, D^* L^{1,2}(W).$$

Use this fact and elliptic regularity to prove the following fact about the space of smooth p-forms:

Hodge Theorem. *On a compact oriented Riemannian manifold (M, g) there is an L^2-orthogonal decomposition*

$$\Omega^p(M) = \mathcal{H}^p \oplus d\Omega^{p-1}(M) \oplus d^*\Omega^{p+1}(M)$$

for each p. Consequently, each DeRham cohomology class is represented by a unique harmonic form, so there is a natural isomorphism $\mathcal{H}^p = H^p(M; \mathbb{R})$.