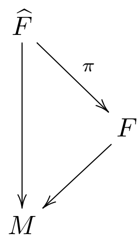
Topics in Gauge Theory

Problem Set 2

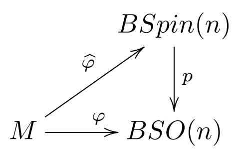
Due Friday, Oct 11

Exerecise 5. Let M be an n-dimensional oriented Riemannian manifold M and let $p_n : Spin(n) \to SO(n)$ be the natural projection. Recall that a spin structure on M is a lift of the $SO(n)$ frame bundle F of T^*M to a principal $Spin(n)$ bundle $\widehat{F}$, that is a 2-fold covering map π so that



$$\pi(fg) = \pi(f)p_n(g) \qquad \forall f \in \widehat{F}, g \in Spin(n)$$

Prove that there exists a spin structure on M if and only if the classifying map $\varphi : M \to BSO(n)$ lifts to a map $\widehat{\varphi} : M \to BSpin(n)$:



Exerecise 6. Let $E, F \to M$ be a hermitian vector bundles on a compact Riemannian manifold M. Prove that if $D : \Gamma(E) \to \Gamma(F)$ is self-adjoint, $(D^* = D)$, the index $D = 0$.

Hint: use the L^2 inner product to show that $coker D \cong (image\, D)^{\perp} \cong \ker D^*$.

Exerecise 7. Let $V \cong \mathbb{R}^n$ be the first fundamental representation of $O(n)$ and let

$$\mathcal{R} = \left\{ \sum R_{ijk\ell}\, e^j \otimes e^i \otimes e^k \otimes e^\ell \;\middle|\; R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij} \right\}$$

be the space of curvature tensors. Show that the vector space of all $O(n)$-equivariant maps

$$\phi : \mathcal{R} \to V \otimes V$$

has dimension 2 and write down a basis of this space.