Topics in Gauge Theory

Problem Set 1

Due Friday, September 20

Exerecise 1. Let $E \to M$ be a vector bundle with hermitian metric $\langle\ ,\ \rangle$ and connection ∇. Fix a path $\gamma : [0,1] \to M$ in M from $p = \gamma(0)$ to $q = \gamma(1)$. Recall that the holonomy along γ is the map

$$\mathcal{H}_\gamma^\nabla : E_p \to E_q$$

defined by $\mathcal{H}_\gamma^\nabla(\xi_0) = \xi(1)$ where $\xi(t)$ is the unique section of E along γ that satisfied $\nabla_{\dot\gamma}\xi = 0$ and $\xi(0) = \xi_0$. Show that $\mathcal{H} = \mathcal{H}_\gamma^\nabla$ is

(a) linear

(b) unitary: $\langle \mathcal{H}\xi, \mathcal{H}\eta\rangle = \langle \xi, \eta\rangle \qquad \forall \xi, \eta \in E_p$, and

(c) gauge-equivariant: $\mathcal{H}_\gamma^{g\cdot\nabla} \ = \ g(q) \circ \mathcal{H}_\gamma^\nabla \circ g(p)^{-1} \quad \forall g \in G.$

Exerecise 2. Given a principal G-bundle $P \to M$ and a representation $\rho : G \to End(V)$ of G on a vector space V, we can form the associated vector bundle

$$E = P \times_\rho V \ = \ \Big\{ (p, v) \in P \times V \ \Big|\ (p, v) \sim (pg, \rho(g^{-1})v) \Big\}$$

(a) Prove that the space of sections of E can be identified with the set of all smooth equivariant maps from P to V:

$$\Gamma(E) \ = \ \Big\{ \phi : P \to V \ \Big|\ \phi(pg) = \rho(g^{-1})\phi(p) \ \ \forall g \in G \Big\}$$

(b) Suppose that $\rho' : G \to End(V')$ is another representation of G and that $\phi : V \to V'$ is a G-equivariant map (i.e. $\phi(\rho(g)v) = \rho'(g)\phi(v)\ \forall g \in G, v \in V$). Prove that ϕ induces a bundle map

$$\Phi : E \to E'$$

where $E = P \times_\rho V$ and $E' = P \times_{\rho'} V'$.

Exerecise 3. Let $\pi : P \to M$ be a principal G-bundle and let

$$ad(P) \ = \ P \times_{Ad} \mathfrak{g}$$

be the vector bundle associated to P by the Adjoint representation $Ad : G \to End(\mathfrak{g})$. Recall Definition 2: A connection on P is a $\mathfrak{g}$-valued 1-form ω on P that satisfies

(i) $\omega(\iota_* A) = A$ for all $A \in \mathfrak{g}$ where $\iota_* : \mathfrak{g} \to TP$ is the map induced by the right G-action on P.

(ii) $(R_g)_*\omega = Ad(g^{-1})\,\omega \quad \forall g \in G.$

(cf. the Kobayahsi-Nomizu handout, page 64). Using this definition, show that the difference of two connection 1-forms is an element of $\Omega^1(adP)$, that is, an $ad(P)$-valued 1-form on M.

It follows that the the space of connections $\mathcal{A}$ on P is an (infinite-dimensional) affine space modeled on $\Omega^1(adP)$.

Exerecise 4. Given complex representations (ρ_1, V) and (ρ_2, W) of a Lie group G, let $Hom^G(V, W)$ denote the vector space of equivariant maps $\phi : V \to W$ (i.e. maps satisfying $\phi(\rho_1(g)v) = \rho_2(g)\phi(v)$ for all $g \in G, v \in V$). *Schur's Lemma* states that If V and W are irreducible then

$$Hom^G(V, W) = \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove Schur's Lemma (Hint: show that $\ker \phi$ and $im\phi$ are G-invariant subspaces).

(b) Let Δ_1 be the 3-dimensional irreducible representation of $G = SU(2)$. Use the Clebsch-Gordon decomposition to show that there is a non-trivial G-equivariant map

$$\Delta_1 \to Hom(\Delta_{\frac{k}{2}}, \Delta_{\frac{\ell}{2}})$$

if and only if $k - \ell = \pm 2$.

(c) The group $Spin(4)$ is isomorphic to $SU(2) \times SU(2)$. Hence, under the isomorphism $Spin(4) = SU^+(2) \times SU^-(2)$, the irreducible complex representations are of the form $\Delta^+_{\frac{k}{2}} \otimes \Delta^-_{\frac{\ell}{2}}$ with the obvious notation. In this notation, the complexification of the standard representation V_1 of $Spin(4)$ on $\mathbb{R}^4$ is $\Delta_{\frac{1}{2}} \otimes \Delta_{\frac{1}{2}}$. Extend part (a) above to characterize the space of non-trivial G-equivariant maps

$$\Delta_1 \to Hom(V, W)$$

where V and W are complex irreducible representations of $G = Spin(4)$.