Geometric Analysis   Problem Set 4

Due Wednesday March 17

Problem (4.1) Let X be the unit circle. Explicitly describe the eigenspaces of

$$\begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \qquad \text{acting on} \qquad \Omega^0_X \oplus \Omega^1_X.$$

For the following problems, recall that, by the spectral theorem, each $\phi \in L^2(V)$ has an L^2-orthogonal "Fourier series" expansion

$$\phi = \sum a_\lambda \phi_\lambda \tag{0.1}$$

where the ϕ_λ are eigenvectors of D^*D. The L^2 norm of ϕ is then

$$\| \phi \|^2 = \sum |a_\lambda|^2 < \infty. \tag{0.2}$$

Problem (4.2) Use the expansion (0.1) to prove the following version of the Poincaré inequality: if $\phi \in L^2$ is L^2-perpendicular to the space of all eigenspaces with eigenvalues $\lambda \leq \Lambda$ then

$$\| \phi \|_{1,2} \leq C \ \| D\phi \|_{0,2} \qquad \text{where} \qquad C^2 = 1/\Lambda.$$

Problem (4.3) Fix $k \geq 0$. Define a norm on $\phi \in \Gamma(V)$ by expanding ϕ as in (0.1) and setting

$$\| \phi \|^2 = \sum_\lambda (1+\lambda)^k \, |a_\lambda|^2.$$

(a) Prove that this norm is equivalent to the $L^{k,2}$ norm.
(b) Define Sobolev spaces of functions $W^{s,2}(M)$ for real numbers $s > 0$.

Problem (4.4) Read the last page of Section 4 of the "Geometry Primer" and answer these questions:

Let $D_t : \Gamma(V) \to \Gamma(W)$, $t \in [0,1]$, be a path of first order simple elliptic operators.

(a) Show that the non-zero spectrum of $D_t^* D_t$ is the same as that of $D_t D_t^*$.

(b) Assuming that the eigenvalues depend continuously on t (they do), use part (a) to show that the index is independent of t.

Problem (4.5) Suppose that $\gamma : S^1 \to M$ is an $W^{1,2}$ weak solution of the geodesic equation $\nabla_T T = 0$. (Here ∇ is the Levi-Civita connection of (M, g) and $T = \dot{\gamma} = \frac{d}{dt}\gamma(t)$ is the tangent vector to the loop γ.)

(a) Write the geodesic equation in local coordinates $\{x^i\}$ on M.

(b) Is the equation linear? Elliptic?

(c) Use the Sobolev inequalities to show that each point $t_0 \in S^1$ has a neighborhood whose image under γ lies in a single coordinate chart of M.

(d) Use bootstrapping to prove that γ is C^∞.