Geometric Analysis   Problem Set 1

Due Monday, January 25

Problem (1.1) Use a partition of unity to prove that the set

$$\text{Metric}(M) = \{\text{all Riemannian metrics on the manifold } M\}$$

is a non-empty convex cone (without vertex) in the vector space $\Gamma(Sym^2(T^*M))$.

Problem (1.2) Let ∇ and ∇' be connections compatible with a metric $\langle\ ,\ \rangle$ on a vector bundle E. Show that:

(a) For any $f \in C^\infty(M)$, $\nabla'' = f\nabla + (1-f)\nabla'$ is also a connection compatible with the metric.

(b) $\nabla - \nabla' = A$ is an $\text{End}(E)$-valued 1-form (i.e., an element of $\Gamma(T^*M \otimes \text{End}(E))$ that is skew-hermitian when E is complex and skew-symmetric when E is real.

(c) Conversely, with ∇ and A as in (b), show that $\nabla' = \nabla + A$ is a connection compatible with the metric.

Note that (b) and (c) show that

$$\mathcal{A} = \{\text{all compatible connections on } E\}$$

is an infinite-dimensional affine space modeled on $\Gamma(T^*M \otimes \text{SEnd}(E))$ where $\text{SkewEnd}(E))$ is the bundle of skew-hermitian endomorphisms of E.

Hint: For (b), use the fact that any $C^\infty(M)$-linear map $\Phi : \Gamma(E) \to \Gamma(F)$ arises in this way from a bundle map $\phi : E \to F$ by composition: $\Phi(f\xi) = f\Phi(\xi)\ \ \forall f \in C^\infty(M)$.

Problem (1.3) Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). In a local coordinate system $\{x^i\}$, we write the metric as

$$\sum g_{ij}\, dx^i \otimes dx^j$$

and define the Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

(a) Show that $\nabla_i = \partial_i + \Gamma^k_{ij}$, i.e. for vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^j \frac{\partial}{\partial x^j}$

$$\nabla_X Y = \sum X^i \left(\frac{\partial}{\partial x^i} + \Gamma^k_{ij} Y^j \right) \frac{\partial}{\partial x^k}.$$

(b) Show that the torsion-free condition implies that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

The components of the Riemannian curvature tensor R are defined by

$$\sum R^i_{jk\ell} \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}$$

(c) Derive the classical expression $R^i_{jkl} = \sum (\partial_k \Gamma^i_{\ell j} - \partial_\ell \Gamma^i_{kj}) + (\Gamma^m_{\ell j} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{\ell m})$

Problem (1.4) Let ∇ and ∇' be two connections on a vector bundle $E \to M$. Write $\nabla' = \nabla + A$ where A is an End(E)-valued 1-form. Show that the curvatures of ∇ and ∇' are related by

$$F^{\nabla'} = F^{\nabla} + d^{\nabla} A + [A, A]$$

where

$$d^{\nabla} : \Gamma(T^*M) \otimes End(E) \to \Gamma(\Lambda^2 T^*M \otimes End(E))$$

is the covariant exterior derivative defined by $d^{\nabla} A(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X)$, and $[A, A]$ is the End(E)-valued 2-form given by $[A, A](X, Y) = A(X)A(Y) - A(Y)A(X)$.

Problem (1.5) Prove the second Bianchi identity: the curvature tensor satisfies

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0.$$