1 Direct Minimization

One can often obtain existence of weak solutions by minimizing an energy function. The "direct method" involves choosing a minimizing sequence and finding a limit. It is useful to codify this process.

Definition. *A function $E : X \to \mathbb{R}$ on a Banach space is* **coercive** *if any sequence $\{x_i\}$ with $E(x_i) < C$ is bounded.*

In practice, this is usually verified by a "coercive estimate" of the form $\| x \|^2 \leq C_1 E(x) + C_2$.

Theorem 1.1 (Direct Minimization) *Suppose that $E : S \to \mathbb{R}$ is a function on a nonempty, weakly closed subset S of a reflexive Banach space X which is*

(a) bounded below,

(b) coercive, and

(c) weakly lower semicontinuous (wlsc).

Then there exists a minimizer $x_0 \in S$ for E.

Proof. Choose a sequence $\{x_1\} \in S$ with $E(x_i) \to E_0 = \inf_{x \in S} E(x)$. Coerciveness implies that the sequence is bounded, so there is a weakly convergent subsequence $x_i \rightharpoonup x_0$ with $x_0 \in S$. But then $E_0 \leq E(x_0) \leq \liminf E(x_i) = E_0$ by weak lower semicontinuity. □

We've seen weak lower semicontinuity in one case: the norm in a Banach space is wlsc. The following proposition provides more examples. Recall that a function E is convex if

$$E(tx + (1-t)y) \;\leq\; tE(x) + (1-t)E(y) \qquad \text{for all } t \in [0,1] \text{ and } x, y \in X.$$

Proposition 1.2 *If E, as in Theorem 1, is a finite sum of continuous convex functions on X and functions continuous with respect to weak convergence, then E is wlsc.*

Proof. The sum of wlsc functions is wlsc, and continuous functions are wlsc, so it suffices to verify that convex functions are wlsc. Suppose that $x_i \rightharpoonup x_0$; after passing to a subsequence we can assume that $E(x_i) \to \liminf E(x_i)$. Let K be the convex hull of $\{x_i\}$. Then x_0 lies in the weak closure of K. But the weak closure of K is the closure of K by the Hahn-Banach Theorem. Thus $x_0 \in \bar{K}$, so there is a sequence $\{y_n = \sum a_{kn} x_k \mid \sum_k a_{kn} = 1\}$ of convex combinations such that $y_i \to x_i$. Using Jensen's inequality we then have

$$E(x_0) \;=\; \lim E(y_n) \;\leq\; \limsup \sum_{k>N} a_{nk} E(x_k) \;\leq\; \limsup E(x_i) \;=\; \liminf E(x_i).$$

□

Here is an easy-to-recognize criterion: if

$$E(x) = \sum B_i(x,x) + L_i(x)$$

where each B_i is a continuous ($|B_i(x,y)| \leq C|x| \cdot |y|$) non-negative symmetric bilinear form and each L_i is the composition of a compact operator $X \to Y$ and a continuous function $Y \to \mathbb{R}$ then E is wlsc.

Example: For fixed $g \in L^2$, the energy function

$$E(f) = \int_M |df|^2 - 2gf$$

is wlsc on $L^{1,2}$ (the second term is the composition of the compact embedding $L^{1,2} \to L^2$ and the bounded linear functional $f \mapsto 2gf$ on L^2).