Geometry Primer

1 Connections and Curvature

This section presents the basics of calculus on vector bundles. It begins with the basic abstract definitions, then gives some concrete geometric examples.

Let E be a (real or complex) vector bundle over a manifold M. There are three levels of geometric structures on E:

- Metrics
- Covariant derivatives
- Second covariant derivatives. These decompose into

  (i) the covariant Hessian (the symmetric part), and

  (ii) the curvature (the skew-symmetric part).

Definition A *metric* on a vector bundle E is a smooth choice of a hermitian inner product on the fibers of E, that is, an $h \in \Gamma(E^* \otimes E^*)$ such that

$$\text{(i)}\ \ h(\alpha, \beta) = \overline{h(\beta, \alpha)} \qquad \forall \alpha, \beta \in \Gamma(E),$$

$$\text{(ii)}\ \ h(\alpha, \alpha) \geq 0 \qquad \forall \alpha \in \Gamma(E) \text{ and } h(\alpha, \alpha) = 0 \text{ iff } \alpha \equiv 0.$$

We will take our hermitian metrics to be conjugate linear in the second variable. When E is a real vector bundle, (i) simply means that h is symmetric.

A metric on the tangent bundle TM is called a *Riemannian metric* on M.

In a local coordinate system $\{x^i\}$ on $U \subset M$ the vector fields $\frac{\partial}{\partial x^i}$ give a basis of the vector space T_xM at each $x \in U$ and the Riemannian metric is given by the symmetric matrix

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

by the formula $g = \sum_i g_{ij}(x)\, dx^i \otimes dx^j$.

Similarly, a *local frame* of E over $U \subset M$ is a set $\{\sigma_\alpha\}$ of sections of E over U such that the vectors $\{\sigma_\alpha(x)\}$ form a basis of the fiber $\pi^{-1}(x)$ at each $x \in U$. Write $\{\sigma^\alpha\} \in \Gamma(E^*)$ for the dual framing (so $\sum_\alpha \sigma^\alpha \cdot \sigma_\beta = \delta^\alpha_\beta$). In such a framing the metric on E is given by the hermitian matrix

$$h_{\alpha\overline{\beta}} = h(\sigma_\alpha, \sigma_\beta)$$

by the formula $h = \sum_\alpha h_{\alpha\beta}\, \sigma^\alpha \otimes \overline{\sigma^\beta}$, and for $\phi = \sum \phi^\alpha \sigma_\alpha$ we have $h(\phi,\phi) = \sum h_{\alpha\overline{\beta}} \phi^\alpha \overline{\phi^\beta}$.

A frame is *orthogonal* or *unitary* if $\{\sigma_1, \dots, \sigma_\ell\}$ is an orthonormal basis for E_x at each x. Local unitary frames always exist (start with any frame and apply the Gram-Schmidt process). In a unitary frame, the metric is simply $h = \sum \sigma^\alpha \otimes \overline{\sigma^\alpha}$, so the coefficients $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ are constant.

An inner product on a vector space V induces inner products on V^*, on the exterior algebra $\Lambda^*(V)$, and on tensor products of these vector spaces. Applying this on each fiber shows that a metric on E induces metrics on E^*, $\Lambda^*(E)$ and on tensor product bundles. Simple examples:

- A metric h on E gives a metric on $E \otimes E$ by the formula

  $$h(\alpha \otimes \beta, \alpha \otimes \beta) = h(\alpha,\alpha)\ h(\beta,\beta) \qquad \text{for } \alpha, \beta \in \Gamma(E)$$

  and one on $\Lambda^2(E)$ by (using the convention $\alpha \wedge \beta = \frac{1}{\sqrt{2}}(\alpha \otimes \beta - \beta \otimes \alpha)$)

  $$h(\alpha \wedge \beta, \alpha \wedge \beta) = h(\alpha,\alpha)\ h(\beta,\beta) - [h(\alpha,\beta)]^2.$$

- A metric h on E gives an identification of E with E^*, and hence gives a metric on E^*. When $E = TM$ this identification is given in local coordinates by

  $$\frac{\partial}{\partial x^i} \mapsto \sum g_{ij} dx^i \quad \text{and} \quad dx^i \mapsto \sum (g^{-1})^{ij} \frac{\partial}{\partial x^j}.$$

  The ij component of the induced metric g^* on T^*M is

  $$g^*(dx^i, dx^j) \;=\; \sum g\left((g^{-1})^{ik} \frac{\partial}{\partial x^k},\ (g^{-1})^{j\ell} \frac{\partial}{\partial x^\ell} \right) \;=\; \sum g_{k\ell} (g^{-1})^{ik} (g^{-1})^{j\ell} \;=\; (g^{-1})^{ij}.$$

A useful and standard convention is to write g_{ij} for the metric and g^{ij} for the components of its inverse, and to omit all summation signs, agreeing that repeated indices are summed. If one uses upper indices on the coordinate 1-forms dx^i and thinks of the coordinate vector fields $\partial/\partial x^i$ as having lower indices, then all formulas are consistent in the sense that all sums are over one upper and one lower index.

Connections

We would next like to define the "directional derivative" of a section $\phi \in \Gamma(E)$. To specify the direction we choose a vector field X; the dirctional derivative should compare the value of ϕ at $x \in M$ with the value at nearby points $x_t = \exp_x(tX)$. But the naive definition

$$\partial_X \phi(x) = \lim_{t \to 0} \frac{\phi(x_t) - \phi(x)}{t}$$

makes no sense because $\phi(x)$ and $\phi(x_t)$ are in different fibers of E and cannot be subtracted. Thus to define a derivative we need an additional geometric structure on E: an isomorphism between nearby fibers. Actually, we need this only infinitesimally. This is what a "connection" does.

There are many definitions of connections. We will start by defining a connection as an operator on sections with the properties expected of a directional derivative.

Definition 1.1 *A covariant derivative (or connection) on E is a bilinear map*

$$\nabla : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$$

that assigns to each vector field X and each $\phi \in \Gamma(E)$ a "covariant directional derivative" $\nabla_X \phi$ satisfying, for each $f \in C^\infty(M)$,

(i) $\nabla_{fX}\phi = f\nabla_X\phi$

(ii) $\nabla_X(f\phi) = (X \cdot f)\phi + f\nabla_X\phi$ *(product rule).*

Given connections on vector bundles E and F we get one on $E \otimes F$ by the product rule:

$$\nabla_X^{E\otimes F}(\phi \otimes \psi) = \nabla_X^E \phi \otimes \psi + \phi \otimes \nabla_X^F \psi, \qquad \phi \in \Gamma(E), \psi \in \Gamma(F).$$

Similarly, a connection on E induces one on E^*: for $\phi \in \Gamma(E), \alpha \in \Gamma(E^*)$, the derivative of the function $\alpha(\phi)$ is, according to the product rule, $X \cdot \alpha(\phi) = (\nabla^{E^*}\alpha)\phi + \alpha(\nabla^E \phi)$, so ∇^{E^*} is defined by

$$(\nabla^{E^*}\alpha)\phi \ = \ X \cdot \alpha(\phi) \ - \ \alpha(\nabla^E \phi).$$

In particular, the metric h can be considered a section of the bundle $E^* \otimes E^*$. Then for $\phi, \psi \in \Gamma(E)$, $h(\phi, \psi)$ is the trace of a section of $E^* \otimes E^* \otimes E \otimes E$ so, again applying the product rule, for any vector field X

$$X \cdot h(\phi, \psi) = (\nabla_X h)(\phi, \psi) + h(\nabla_X \phi, \psi) + h(\phi, \nabla_X \psi). \tag{1.1}$$

Definition 1.2 *A connection ∇ is compatible with the metric h on E if $\nabla h = 0$.*

Each vector bundle with metric admits a compatible connection (see below). The difference of two connections is an End(E)-values 1-form (from the definition $(\nabla - \nabla')_X\phi$ is $C^\infty(M)$-linear in X and ϕ). Conversely, given a compatible connection ∇ and $A \in \Gamma(T^*M \otimes \mathrm{End}(E))$,

$$\nabla' = \nabla + A$$

is a connection, which is compatible iff $A \in \Gamma(T^*M \otimes \mathrm{SkewEnd}(E))$. Thus the space of all compatible connections is an infinite-dimensional affine space.

Henceforth we will always assume that the connection is compatible with the metric, and will write the metric $h(\alpha, \beta)$ as $\langle \alpha, \beta \rangle$. Then (1.1) becomes

$$X \cdot \langle \alpha, \beta \rangle = \langle \nabla_X \alpha, \beta \rangle + \langle \alpha, \nabla_X \beta \rangle.$$

In a local framing $\{\sigma_\alpha\}$ over a coordinate patch $\{x^i\}$, the covariant derivative determines connection forms $\omega^\alpha_{\beta i}$ by

$$\nabla_{\frac{\partial}{\partial x^i}} \phi^\alpha = \sum \omega^\alpha_{\beta i}\, \phi^\beta.$$

For a general section $\phi = \sum \phi^\alpha \sigma_\alpha$ and vector field $X = \sum X^i \frac{\partial}{\partial x^i}$ we then have

$$\nabla_X \phi = \sum X^i \nabla_{\frac{\partial}{\partial x^i}} (\phi^\alpha \sigma_\alpha) = \sum X^i \left(\frac{\partial \phi^\alpha}{\partial x^i} + \omega^\alpha_{\beta i} \phi^\beta \right) \sigma_\alpha. \tag{1.2}$$

Thus the connection forms give the difference between the covariant derivative and the ordinary derivative in the framing. Note that it is the covariant derivative that is intrinsic; when we change framings the operators $\frac{\partial}{\partial x^i}$ and the connection forms both change.

We can now prove existence. Let $\{U_\gamma, \rho_\gamma\}$ be a partition of unity where each U_γ is a local coordinate chart over which E is trivialized by a local frame $\{\sigma_\alpha\}$. For vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ supported in one U_γ set

$$\nabla_X \phi = \begin{cases} \sum X^i \dfrac{\partial \phi}{\partial x^i} & \text{on } U_\gamma \\ 0 \quad \text{outside } U_\gamma \end{cases}$$

and for general vector fields set $\nabla_X \phi = \sum \nabla_{\rho_\gamma} \phi$. It is easily verified that this defines a connection. If the frame $\{\sigma_\alpha\}$ is unitary, then the coefficients of the metric H are constant on each U_γ. Consequently $\nabla h = 0$, so the connections is compatible with h.

In the special case where E is the tangent bundle we can impose an additional requirement on the connection. A connection ∇ on TM is called *torsion-free* or *symmetric* if

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad \text{for all } X, Y \in \Gamma(TM).$$

The following fact, often called the Fundamental Lemma of Riemannian Geometry, shows that these two conditions determine a connection.

Lemma 1.3 *On a manifold with Riemannian metric g, there is a unique connection ∇ on TM, the "Levi-Civita connection", that is (a) compatible with the metric, and (b) torsion free.*

Proof. For any three vector fields X, Y, Z, condition (a) requires that

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Computing $X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y)$ using this formula and condition (b) yields

$$\begin{aligned} 2g(\nabla_X Y, Z) \;&=\; X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) - g(Z, [X, Y]). \end{aligned} \tag{1.3}$$

Both sides are linear in Z and g is non-degenerate. Uniqueness follows because the righthand side depends only on g. Conversely, requiring that this hold for all Z defines $\nabla_X Y$. One checks directly that this defines a torsion free connection with $\nabla g = 0$. $\square$

In local coordinates on a Riemannian manifold we can write the metric as $\{g_{ij}\}$. Taking coordinate vector fields $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$, we have $[X, Y] = 0$ and, from (1.3)

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \;=\; \sum_l \Gamma^\ell_{ij} \frac{\partial}{\partial x^\ell}$$

where Γ^l_{ij} are the Christoffel symbols

$$\Gamma^l_{ij} = \sum \frac{1}{2} g^{lk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

For general vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^j \frac{\partial}{\partial x^j}$ we have, as in (1.2),

$$\nabla_X Y = \sum X^i \left(\frac{\partial Y^j}{\partial x^i} + Y^k \Gamma^j_{ik} \right) \frac{\partial}{\partial x^j}.$$

Again, the Christoffel symbols and the operators $\frac{\partial}{\partial x^i}$ depend on the coordinates, but the covariant derivative does not.

A connection (on any vector bundle) gives a way of parallel transporting sections along curves. Fix a smooth curve $\gamma : [a, b] \to M$ from $x = \gamma_a$ to $y = \gamma_b$ and a vector ξ in the fiber E_x at x. We can then solve the initial value problem

$$\nabla_T \xi_t = 0 \qquad \text{with} \qquad \xi_a = \xi \tag{1.4}$$

where $T = \dot{\gamma}$ is the tangent vector to $\gamma(t)$. Evaluating the solution at $t = b$ yields a vector $\xi_b \in E_y$. This process defines a linear map

$$P_\gamma : E_x \to E_y$$

called the *parallel transport of* ξ *along* γ.

Remark 1.4 To show the existence and uniqueness of solutions of (1.4), cover γ with finitely many coordinate patches $\{U_i\}$ on which E is trivialized. In the trivialization on U_i the above equation has the form

$$\sum T^i \left(\frac{\partial \xi^\alpha}{\partial x^i} + \xi^\beta \omega^\alpha_{\beta i} \right) = 0. \tag{1.5}$$

Hence in each patch we can begin at $\gamma_c \in U_{i-1} \cap U_i$ and, by the fundamental theorem of ODEs, find a unique solution for $t \in [c, d]$ where $\gamma_d \in U_i \cap U_{i+1}$.

Having integrated, we can differentiate again and see that the connection is infinitestimal parallel transport

$$(\nabla_X \xi)_p \;=\; \lim_{t \to 0} \frac{P_{-t} \xi(p_t) - \xi(p)}{t} \tag{1.6}$$

where P_{-t} denotes parallel transport along the path $x_t = \exp(tX)$ from p_t back to p.

Proof. Along $\gamma(t) = \exp(tX)$ the solution to the parallel transport equation (1.4) can be written in local frame around $p \in M$ as $\xi = \sum \xi^\alpha(t) \sigma_\alpha$. The Taylor series of the coefficients is

$$\xi^\alpha(t) \;=\; \xi^\alpha(0) + tX^i \frac{\partial \xi^\alpha}{\partial x^i} + O(t^2)$$

and, since ξ satisfies the parallel transport equation (1.5), we have

$$P_t(\eta^\alpha) \;=\; \eta^\alpha(0) - tX^i \omega^\alpha_\beta \eta^\beta + O(t^2).$$

Replacing t by $-t$ and η by $\xi^\alpha(t)$, we see that the RHS of (1.6) is

$$\lim_{t\to 0} \frac{1}{t}\left(\xi^\alpha(0) + tX^i\frac{\partial \xi^\alpha}{\partial x^i} + tX^i\omega^\alpha_\beta\xi^\beta - \xi^\alpha(0)\right) \;=\; X^i\left(\frac{\partial \xi^\alpha}{\partial x^i} + \omega^\alpha_\beta\xi^\beta\right)\sigma_\alpha \;=\; (\nabla_X\xi)_p\,.$$

□

Caution While the limit (1.6) looks very similar to the limit defining the Lie derivative $\mathcal{L}_X Y$, the two are unrelated. In particular, parallel transport is dependent on the choice a Riemannian metric, while the Lie derivative is defined solely in terms of the vector fields X and Y.

The definition of compatibility has the following two important consequences.

Lemma 1.5 *When the connection is compatible with the metric,*

1. *Parallel transport is an isometry, and*
2. *We have the pointwise inequality*
$$|d|\xi|| \le |\nabla\xi|.$$

Proof. (1) Given a path $\gamma(t)$ and vectors ξ_0, η_0 in the fiber of E at $\gamma(0)$, extend ξ_0, η_0 to vector fields ξ_t, η_t that are parallel along γ. Then for all t we have

$$\frac{d}{dt}\langle\xi_t, \eta_t\rangle \;=\; T\cdot\langle\xi_t,\eta_t\rangle \;=\; \langle\nabla_T\xi_t, \psi_t\rangle + \langle\xi_t, \nabla_T\eta_t\rangle \;=\; 0.$$

Thus inner products are preserved by parallel transport.

(2) For a quick proof, note that the equation $df^2 = 2f df$ gives $d|\xi|^2 = 2|\xi|d|\xi|$, while compatibility with the metric gives $|d|\xi|^2| = |2\langle\xi,\nabla\xi\rangle| \le 2|\xi||\nabla\xi|$. Combining these gives the inequality in (2).

For a more enlightening proof, use polar coordinates in the fiber: on the set Ω where $\phi \neq 0$, set $\phi = \frac{\xi}{|\xi|}$. Then $\xi = |\xi|\,\phi$ and differentiating the equation $|\phi|^2 = 1$ shows that $2\langle\phi,\nabla\phi\rangle = 0$. Hence

$$\begin{aligned}|\nabla\xi|^2 \;=\; |\nabla(|\xi|\,\phi)|^2 \;=\; |d|\xi|\,\phi + |\xi|\cdot\nabla\phi\;|^2 \;&=\; |d|\xi||^2\,|\phi|^2 + 2|\xi|\,d|\xi|\,\langle\phi,\nabla\phi\rangle + |\xi|^2|\nabla\phi|^2\\ &=\; |d|\xi||^2 + |\xi|^2|\nabla\phi|^2\\ &\ge\; |d|\xi||^2,\end{aligned}$$

so (2) holds on Ω and hence everywhere. □

Covariant Second Derivatives

A connection on E

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E)$$

together with the Levi-Civita connection on T^*M gives a connection on $T^*M \otimes E$. The composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E)$$

is the *covariant second derivative*. Since

$$\begin{aligned} \nabla_X(\nabla_Y \xi) \;=\; \nabla_X(\nabla\xi(Y))) &= (\nabla_X \nabla\xi)(Y) + \nabla\xi(\nabla_X Y) \\ &= (\nabla^2\xi)(X,Y) + (\nabla\xi)(\nabla_X Y) \end{aligned}$$

the covariant second derivative is given by

$$(\nabla^2\xi)(X,Y) = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi \qquad \text{for } X, Y \in \Gamma(TM),\ \xi \in \Gamma(E).$$

This expression is $C^\infty(M)$–bilinear in both X and Y.

Taking minus the trace of the covariant second derivative (in analogy with $d^*d = -\sum \partial_i\partial_i$ in euclidean space) gives a second order operator

$$-\mathrm{tr}\, \nabla^2 : \Gamma(E) \to \Gamma(E)$$

called the *trace Laplacian*. It is the same as the composition of ∇ with its adjoint ∇^* (exercise), and is given in a local orthonormal frame $\{e_i\}$ by

$$-\mathrm{tr}\, \nabla^2\xi = -\sum \left(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_i} e_i\right)\xi.$$

Unlike second derivatives in euclidean space, covariant second derivatives do not commute. The expression that measures the failure to commute

$$\begin{aligned} (\nabla^2\xi)(X,Y) - (\nabla^2\xi)(Y,X) \;&= \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi - \nabla_Y\nabla_X\xi + \nabla_{\nabla_Y X}\xi \\ &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \end{aligned}$$

$C^\infty(M)$– linear in X, Y *and* ξ. This last fact, which is easily verified, means that the difference of these second order operators is a zeroth order operator, i.e. a tensor.

Definition The curvature of a connection ∇ is the tensor $F \in \Gamma(T^*M \otimes T^*M \otimes \mathrm{End}(E))$ given, for $X, Y \in \Gamma(TM)$, by

$$F(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \tag{1.7}$$

When ∇ is the Levi-Civita connection of a Riemannian metric g, the curvature is denoted $R(X,Y)$ and is called the *Riemannian curvature* of (M, g).

Proposition 1.6 (Symmetries of the curvature) *Let ∇ be a connection on $E \to M$ compatible with a metric $\langle\ ,\ \rangle$. Then for all vector fields X, Y, Z and sections $\xi, \eta \in \Gamma(E)$,*

(a) $F(X,Y) = -F(Y,X)$

(b) $\langle F(X,Y)\xi, \xi\rangle = -\langle \xi, F(X,Y)\xi\rangle$

(c) $(\nabla_X F)(Y,Z) + (\nabla_Y F)(Z,X) + (\nabla_Z F)(X,Y) = 0$

When $E = TM$, the Riemannian curvature R has an additional symmetry:

(d) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Properties (a) and (b) show that the curvature can be considered as a 2-form with values in the bundle of skew-hermitian (skew-symmetric in the real case) endomorphisms of E, that is

$$F \in \Gamma(\Lambda^2(T^*M) \otimes \mathrm{SkewEnd}(E))$$

In (c) we are using the connection on this bundle obtained from the Levi-Civita connection on T^*M and the given one on E. Properties (c) and (d) are called, respectively, the second and first Bianchi identities.

Proof. Symmetry (a) is obvious from the definition of F. For (b), note that

$$\begin{aligned} \langle \nabla_X \nabla_Y \xi, \xi\rangle &= X \cdot \langle \nabla_Y \xi, \xi\rangle - \langle \nabla_Y \xi, \nabla_X \xi\rangle \\ &= X \cdot Y \cdot \langle \xi, \xi\rangle - X \cdot \langle \xi, \nabla_Y \xi\rangle - Y \cdot \langle \xi, \nabla_X \xi\rangle + \langle \xi, \nabla_Y \nabla_X \xi\rangle. \end{aligned}$$

Hence

$$\begin{aligned} \langle F(X,Y)\xi, \xi\rangle &= \langle (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\xi, \xi\rangle \\ &= (X \cdot Y - Y \cdot X - [X,Y]) \cdot \langle \xi, \xi\rangle + \langle \xi, (\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[Y,X]})\xi\rangle. \end{aligned}$$

Then (b) follows after noting that $[X,Y]f = XYf - YXf$ for $f \in C^\infty(M)$.

The remaining two symmetries follow from the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \forall X, Y, Z \in \Gamma(TM).$$

(The proof is straightforward: using $[X,Y] = XY - YX$ the lefthand side expands to a sum of 12 terms, which cancel.) For (d) we expand $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y$ using the definition (1.7) of curvature and the fact that the Levi-Civita connection is torsion-free. The result is

$$\begin{aligned} (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) &+ (\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X) \\ &+ (\nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y) \\ &= (\nabla_X([Y,Z]) - \nabla_{[Y,Z]} X) + (\nabla_Y([Z,X]) - \nabla_{[Z,Y]} Y) + (\nabla_Z([X,Y]) - \nabla_{[X,Y]} Z) \\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \end{aligned}$$

The proof of (c) is similar. $\square$

Notice that each of the equations in Proposition 1.6 is tensorial, that is, linear over $C^\infty(M)$ in each of their variables. To prove tensorial formulas, it is sufficient to fix an (arbitary) point p

and verify the formula at p for the basis vectors of some trivialization. Often, the proof can be considerably shortened by a clever choice of trivialization. As an example, here is a second proof of formula (d) of Proposition 1.6.

Proof. Fix $p \in M$ and local coodinates $\{x^i\}$ around p. It suffices to verify (d) for the basis vector fields $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$. For these, we have $[X,Y] = [X,Z] = [Y,Z] = 0$, so by the definition of curvature, expression (d) is

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y.$$

But the connection is torsion-free, so the fact that $[X,Y] = 0$ implies that $\nabla_X Y = \nabla_Y X$; similarly $\nabla_X Z = \nabla_Z X$ and $\nabla_Y Z = \nabla_Z Y$. Hence the 6 terms above cancel in pairs, leaving 0. $\square$

Exercises

(1.1) Use a partition of unity to prove that the set

$$\text{Metric}(M) = \{\text{all Riemannian metrics on the manifold } M\}$$

is a non-empty convex cone (without vertex) in the vector space $\Gamma(Sym^2(T^*M))$.

(1.2) Let ∇ and ∇' be connections compatible with a metric $\langle\ ,\ \rangle$ on a vector bundle E. Prove:

(a) For any $f \in C^\infty(M)$, $\nabla'' = f\nabla + (1-f)\nabla'$ is a connection compatible with the metric.

(b) $\nabla - \nabla' = A$ is an End(E)-valued 1-form (i.e., an element of $\Gamma(T^*M \otimes \text{End}(E))$ that is skew-hermitian when E is complex and skew-symmetric when E is real.

(c) Conversely, with ∇ and A as in (b), show that $\nabla' = \nabla + A$ is a connection compatible with the metric.

Note that (b) and (c) show that

$$\mathcal{A} = \{\text{all compatible connections on } E\}$$

is an infinite-dimensional affine space modeled on $\Gamma(T^*M \otimes \text{SkewEnd}(E))$ where SkewEnd(E) is the bundle of skew-hermitian endomorphisms of E.

Hint: For (b), use the fact that any $C^\infty(M)$-linear map $\Phi : \Gamma(E) \to \Gamma(F)$ arises in this way from a bundle map $\phi : E \to F$ by composition: $\Phi(f\xi) = f\Phi(\xi)\ \ \forall f \in C^\infty(M)$.

(1.3) Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). In a local coordinate system $\{x^i\}$, we write the metric as

$$g = \sum g_{ij}\ dx^i \otimes dx^j$$

and define the Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum\ \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

(a) Show that $\nabla_i = \partial_i + \Gamma_{ij}^k$, i.e. for vector fields $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^j \frac{\partial}{\partial x^j}$

$$\nabla_X Y = \sum X^i \left(\frac{\partial}{\partial x^i} + \Gamma_{ij}^k Y^j \right) \frac{\partial}{\partial x^k}.$$

(b) Show that the torsion-free condition implies that $\Gamma_{ij}^k = \Gamma_{ji}^k$.

The components of the Riemannian curvature tensor R are defined by

$$\sum R_{jk\ell}^i \frac{\partial}{\partial x^i} = R\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell} \right) \frac{\partial}{\partial x^j}$$

(c) Derive the classical expression $R_{jkl}^i = \sum (\partial_k \Gamma_{\ell j}^i - \partial_\ell \Gamma_{kj}^i) + (\Gamma_{\ell j}^m \Gamma_{km}^i - \Gamma_{kj}^m \Gamma_{\ell m}^i)$

(1.4) Let ∇ and ∇' be two connections on a vector bundle $E \to M$. Write $\nabla' = \nabla + A$ where A is an End(E)-valued 1-form. Show that the curvatures of ∇ and ∇' are related by

$$F^{\nabla'} = F^{\nabla} + d^{\nabla} A + [A, A]$$

where $d^{\nabla} : \Gamma(T^*M) \otimes End(E) \to \Gamma(\Lambda^2 T^*M \otimes End(E))$ is the covariant exterior derivative defined by

$$d^{\nabla} A(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X),$$

and $[A, A]$ is the End(E)-valued 2-form given by $[A, A](X, Y) = A(X)A(Y) - A(Y)A(X)$.

(1.5) Prove the second Bianchi identity: the curvature satisfies (c) of Proposition 1.6.