A DIRECT EXISTENCE PROOF FOR THE VORTEX EQUATIONS OVER A COMPACT RIEMANN SURFACE

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ABSTRACT

We give a direct proof of an existence theorem for the vortex equations over a compact Riemann surface, exploiting the interpretation of these equations in terms of moment maps.

1. *The vortex equations*

In this paper we shall describe a direct existence proof for the vortex equations over a compact Riemann surface. These equations are a straightforward generalization of the vortex equations on $\mathbb{R}^2$ which were introduced in 1950 by Ginzburg and Landau [**10**] in the theory of superconductivity. From a geometric point of view, they correspond to the absolute minima of the Yang–Mills–Higgs functional. This is defined for a unitary connection A on a hermitian line bundle over $\mathbb{R}^2$ and a smooth section ϕ of that bundle as

$$\mathrm{YMH}(A,\phi) = \int_{\mathbb{R}^2} |F_A|^2 + |d_A\phi|^2 + \tfrac{1}{4}(1-|\phi|^2)^2,$$

where F_A is the curvature of A and $d_A\phi$ is the covariant derivative of ϕ.

If we regard $\mathbb{R}^2$ as the complex plane, we may decompose with respect to the complex structure to obtain $d_A = d'_A + d''_A$. Then, by integration by parts, we can show that YMH is bounded below by $2\pi d$, where d is an integer called the *vortex number*, and that this minimum is attained if and only if

$$\left.\begin{aligned} d''_A\phi &= 0, \\ F_A &= \tfrac{1}{2}*(1-|\phi|^2). \end{aligned}\right\}$$

These equations are invariant under gauge transformations, and the moduli space of solutions is described by the basic existence theorem of Jaffe and Taubes [**14**]. They proved that given d points $x_i \in \mathbb{R}^2$ (possibly with multiplicities), there exists a solution to the vortex equations, unique up to gauge equivalence, with $\phi(x_i) = 0$. This means that the moduli space of *vortices* is the space of unordered d-tuples $S^d\mathbb{C}$. But an element of this space can be thought of as the set of zeros of a monic polynomial

$$p(z) = z^d + a_d z^{d-1} + \ldots + a_1.$$

Hence the moduli space is just the vector space $\mathbb{C}^d$ parametrizing all such polynomials.

We shall study the following more general situation. Let X be a compact Riemann surface endowed with a metric having Kähler form ω. Let L be a smooth complex line bundle over X, with a fixed hermitian metric h. Denote by $\mathscr{A}$ the space of unitary connections on (L, h), and by $\Omega^0(L)$ the space of smooth sections of L.

Received 5 February 1992; revised 22 October 1992.

1991 *Mathematics Subject Classification* 53C07.

As in the $\mathbb{R}^2$ case, the *Yang–Mills–Higgs* functional $\mathrm{YMH}: \mathscr{A} \times \Omega^0(L) \to \mathbb{R}$ is defined by

$$\mathrm{YMH}(A, \phi) = \|F_A\|^2 + \|d_A \phi\|^2 + \tfrac{1}{4}\| \, |\phi|_h^2 - 1 \|^2, \tag{1}$$

where $\| \ \|$ denotes the L^2 norm, $F_A \in \Omega_X^2$ is the curvature of A, $d_A \phi \in \Omega^1(L)$ is the covariant derivative of ϕ, and $|\phi|_h$ is the norm of ϕ with respect to h.

This functional is invariant under the standard action of the gauge group $\mathscr{G}$ of unitary transformations of (L, h), so it defines a functional on the space $(\mathscr{A} \times \Omega^0(L))/\mathscr{G}$.

The integration by parts in the $\mathbb{R}^2$ case is replaced here by the use of the Kähler identities (see [2, 7] for details) to rewrite the functional as

$$\mathrm{YMH}(A, \phi) = 2\|d_A'' \phi\|^2 + \left\| \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2} \right\|^2 + 2\pi d,$$

where d_A'' is the $(0, 1)$ part of the connection, $\Lambda F_A \in \Omega_X^0$ is the contraction of F_A with the Kähler form, and the integer d is the degree of L, that is, its first Chern class.

We then conclude that the functional YMH is bounded below by $2\pi d$, and that this lower bound is attained at $(A, \phi) \in \mathscr{A} \times \Omega^0(L)$ if and only if

$$\left.\begin{aligned} d_A'' \phi &= 0, \\ \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2} &= 0. \end{aligned}\right\}$$

Note that the first equation simply says that ϕ is holomorphic with respect to the holomorphic structure on L induced by $A \in \mathscr{A}$.

Due to the compactness of X, we may integrate the second equation to find an obstruction to its solution, given by the condition

$$d < \frac{\mathrm{Vol}(X)}{4\pi}.$$

However, we may overcome this obstruction by introducing a positive parameter τ in the Yang–Mills–Higgs functional. The modified functional YMH_τ is obtained by substituting

$$\tfrac{1}{4}\| \, |\phi|_h^2 - \tau \|$$

for the third term in (1). Our new functional can be rewritten as

$$\mathrm{YMH}_\tau(A, \phi) = 2\|d_A'' \phi\|^2 + \left\| \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau \right\|^2 + 2\pi\tau d,$$

showing that $2\pi\tau d$ is a bound attained if and only if

$$\left.\begin{aligned} d_A'' \phi &= 0, \\ \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau &= 0. \end{aligned}\right\} \tag{2}$$

These equations are called the *τ-vortex equations*. A necessary condition for the existence of solutions is $d < \tau \mathrm{Vol}(X)/4\pi$; we shall show below that this condition is also sufficient. More precisely, we have the following.

THEOREM. *Let X be a compact Riemann surface equipped with a metric. Let L be a C^∞ line bundle of degree $d > 0$ with a fixed hermitian metric h. Let $D = \sum_{i=1}^{d} x_i$ be an effective divisor of degree d, and consider $\tau > 0$. Then there exists a smooth solution, unique up to gauge equivalence, of equations* (2) *if and only if*

$$d = \deg(L) < \frac{\tau \operatorname{Vol}(X)}{4\pi}. \tag{3}$$

Moreover, this solution is such that $(L, d_A'') = [D]$, the holomorphic bundle determined by D, and the set of zeros of ϕ is the divisor D.

This theorem is completely analogous to the result of Jaffe and Taubes **[14]**, although the parameter τ has no equivalent there.

There are already several proofs of this theorem, in fact of a generalization of it to any compact Kähler manifold. Two of them are due to Bradlow. In **[2]**, he gives a proof by reducing the vortex equations to a differential equation, already studied by Kazdan and Warner **[15]**, of the type

$$-\Delta u + he^u - c = 0,$$

for a real function u on X, where h is a smooth function on X, not identically zero, and c is a real constant. His second proof is a particular case of a more general existence theorem for the vortex equations on a vector bundle of arbitrary rank **[3]**. He uses here methods analogous to those used by Simpson **[17]** in dealing with similar equations on Higgs bundles. In **[7]**, the author has given another proof based on dimensional reduction arguments: namely, we have shown that the vortex equations are a dimensional reduction of the Hermitian–Einstein equation for a connection on a rank two bundle over $X \times \mathbb{CP}^1$. The existence of solutions is then related to the notion of stability for a holomorphic bundle via the theorem of Donaldson, Uhlenbeck and Yau **[5, 6, 20]**.

2. *The proof*

We shall model our proof on that of Hitchin for the self-duality equations **[13]**, which in turn is modelled on Donaldson's proof of the theorem of Narasimhan and Seshadri **[4]**. The key fact that we shall exploit is that the equation

$$\Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0$$

can be regarded as a moment map equation. So before starting our proof, let us pause to recall the basic definitions of moment maps.

Let (M, ω) be a symplectic manifold (for example, a Kähler manifold with its Kähler form) and let G be a Lie group acting on M symplectically, that is, preserving the symplectic form. Then if X is a vector field generated by the action, the Lie derivative $L_X \omega$ vanishes. Now for ω, as for any differential form,

$$L_X \omega = i(X)\, d\omega + d(i(X)\, \omega);$$

hence $d(i(X)\,\omega) = 0$. So, if $H^1(M, \mathbb{R}) = 0$, there exists a function $\mu_X \colon M \to \mathbb{R}$ such that

$$d\mu_X = i(X)\, \omega.$$

As X ranges over the set of vector fields generated by the elements of the Lie algebra $\mathfrak{g}$ of G, these functions can be chosen, under quite general conditions, to fit together to give a map to the dual of the Lie algebra, $\mu\colon M \to \mathfrak{g}^*$, defined by

$$\langle \mu(x), A \rangle = \mu_{\tilde{A}}(x),$$

where $\tilde{A}$ is the vector field generated by $A \in \mathfrak{g}$. There is a natural action of G on both sides, and an ambiguity of a constant in the choice of μ_X. If this can be adjusted so that μ is G-equivariant, that is, compatible with both actions, then μ is called a *moment map* for the action of G on M. The only remaining ambiguity in the choice of μ is the addition of a constant abelian character in $\mathfrak{g}^*$.

A valuable feature of the moment map is that it gives rise to new symplectic manifolds. More precisely, suppose that G acts freely and discontinuously; then

$$\mu^{-1}(0)/G$$

is a symplectic manifold of dimension $\dim M - 2\dim G$. This is the *Marsden–Weinstein quotient* of a symplectic manifold by a group (see **[12, 16]**, for instance).

We shall be concerned with moment maps defined on infinite-dimensional spaces. The first of these was constructed by Atiyah and Bott **[1]**. Let L be a C^∞ complex line bundle over a compact Riemann surface X. Fix a hermitian metric h on L. The space of unitary connections can be identified with $\mathscr{C}$, the space of holomorphic structures on L. On $\mathscr{C}$ there is an inner product

$$\langle \alpha, \beta \rangle = \int_X (\alpha \wedge \beta^*)$$

for $\alpha, \beta \in T_{\bar\partial_L}\mathscr{C} \cong \Omega^{0,1}(X)$. This inner product makes $\mathscr{C}$ a Kähler manifold with Kähler form

$$\omega(\alpha,\beta) = i(\langle \alpha,\beta\rangle - \langle \beta,\alpha\rangle).$$

The standard action of the unitary gauge group $\mathscr{G}$ preserves this Kähler form, and the moment map for the action is given, up to addition of a constant element of the centre, by

$$A \longmapsto \Lambda F_A.$$

Now consider the space of sections $\Omega^0(L)$. This is also a Kähler manifold, with metric given by

$$\langle \psi, \eta \rangle = \int_X (\psi,\eta)_h \quad \text{for } \psi,\eta \in \Omega^0(L).$$

The symplectic form that we shall consider on $\Omega^0(L)$ is

$$\omega(\psi,\eta) = \frac{i}{2}(\langle \psi,\eta\rangle - \langle \eta,\psi\rangle).$$

The action of the gauge group $\mathscr{G}$ is symplectic, and it is easy to see that the moment map is given by $\phi \mapsto -\frac{1}{2}i|\phi|_h^2$. We may combine the moment maps on $\mathscr{A}$ and $\Omega^0(L)$ to obtain the moment map

$$\Lambda F_A - \frac{i}{2}|\phi|_h^2$$

for the symplectic action of $\mathscr{G}$ on $\mathscr{A} \times \Omega^0(L)$.

We shall wish in future to restrict this moment map to the Kähler submanifold

$$\mathcal{N} = \{(A,\phi)\in\mathcal{A}\times\Omega^0(L)\,|\,\phi\not\equiv 0 \text{ and } d''_A\phi = 0\},$$

since pairs (A,ϕ) in $\mathcal{N}$ satisfy the first equation of our system of vortex equations.

In order to introduce the parameter τ in the moment map, consider the symplectic action on $\mathcal{N}$ of the subgroup $U(1)\subset\mathcal{G}$ of constant unitary transformations, acting trivially on A and by multiplication on ϕ. The moment map for this action, $\tilde{\mu}\colon\mathcal{N}\to \mathrm{u}(1)$, is given by

$$\tilde{\mu}(A,\phi) = -\frac{i}{2\,\mathrm{Vol}(X)}\int_X |\phi|_h^2.$$

The range of the moment map $\tilde{\mu}$ is $\mathrm{u}(1) = i\mathbb{R}^-$, where $\mathbb{R}^- = (-\infty,0)$. So if $c\in i\mathbb{R}^-$, we can consider the symplectic quotient $\tilde{\mathcal{N}} = \tilde{\mu}^{-1}(c)/U(1)$. The group $\tilde{\mathcal{G}} = \mathcal{G}/U(1)$ acts symplectically on $\tilde{\mathcal{N}}$, and the moment map μ for this action is given by

$$\begin{aligned}\mu(A,\phi) &= \Lambda F_A - \frac{i}{2}|\phi|_h^2 - \frac{1}{\mathrm{Vol}(X)}\left(\int_X \Lambda F_A - \frac{i}{2}\int_X |\phi|_h^2\right)\\ &= \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{2\pi i d}{\mathrm{Vol}(X)} - c.\end{aligned}$$

We shall be able to solve this equation for any $c\in i\mathbb{R}^-$. Writing

$$\frac{i}{2}\tau = \frac{2\pi i d}{\mathrm{Vol}(X)} - c,$$

then $c\in i\mathbb{R}^-$ is equivalent to $d < \tau\,\mathrm{Vol}(X)/4\pi$, and

$$\mu(A,\phi) = \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau.$$

The second equation of our system of vortex equations is then equivalent to $\mu(A,\phi) = 0$.

The complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts on $\mathcal{A}$ by the action obtained by identifying $\mathcal{A}$ with the space of holomorphic structures on L and on $\Omega^0(L)$ by multiplication. This induces an action on $\mathcal{N}$. We can identify the quotient space $\mathcal{N}/\mathcal{G}^{\mathbb{C}}$ with the space of *effective divisors* of degree d, that is, the d-fold symmetric product of the Riemann surface S^dX. This follows from the very standard facts that a holomorphic line bundle is the line bundle of an effective divisor if and only if it has a non-trivial holomorphic section, and that, moreover, the divisor is given by the zeros of this holomorphic section (see **[11]**, for example).

However, we clearly can also identify the space of effective divisors with the quotient $\tilde{\mathcal{N}}/\tilde{\mathcal{G}}^{\mathbb{C}}$, where $\tilde{\mathcal{G}}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}/\mathbb{C}^*$. The action of $\tilde{g}\in\tilde{\mathcal{G}}^{\mathbb{C}}$ on $\tilde{\mathcal{N}}$ is given by choosing a lifting $g\in\mathcal{G}^{\mathbb{C}}$ that leaves the L^2-norm of the Higgs field fixed, that is, $\|g\phi\|_{L^2} = \|\phi\|_{L^2}$. Clearly, this extends the action of $\tilde{\mathcal{G}}$.

We shall solve the equation $\mu(A,\phi) = 0$ by considering the orbit of a representative of the divisor D in $\tilde{\mathcal{N}}$ under the complex group $\tilde{\mathcal{G}}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}/\mathbb{C}^*$. We shall find a minimum for $\|\mu(A,\phi)\|_{L^2}^2$ on this orbit. Since $\tilde{\mathcal{G}}^{\mathbb{C}}$ acts freely, we shall produce a solution to the equation $\mu(A,\phi)=0$. To see this, suppose that the symplectic manifold M considered at the beginning of our discussion on moment maps is a Kähler manifold, and that the group G acts by isometries as well as symplectically. We choose an invariant positive definite inner product on $\mathfrak{g}$, allowing us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Suppose that G has a complexification $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}}=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$. Then the action of G can be extended to an action of $G^{\mathbb{C}}$. This action preserves the complex structure of M, but not necessarily the metric or symplectic structure. Now consider the function $f\colon M\to\mathbb{R}$ defined by $f(x)=|\mu(x)|^2$. The gradient vector field $\operatorname{grad} f$ at a point $x\in M$ satisfies

$$\begin{aligned}\langle \operatorname{grad} f, X\rangle &= 2\langle \mu(x), d\mu_x(X)\rangle \\ &= 2\langle I\widetilde{\mu(x)}, X\rangle, \end{aligned} \tag{4}$$

where I denotes the complex structure on TM_x, and $\widetilde{\mu(x)}$ is the vector field generated by $\mu(x)$ evaluated at x. Hence

$$\operatorname{grad}_x f = 2I\widetilde{\mu(x)}.$$

By (4), the gradient lines are contained in the orbits of $G^{\mathbb{C}}$. Let Γ be such an orbit; then the critical points of the restriction of f to Γ are also critical points of f on M. If x is a critical point, then $\widetilde{\mu(x)}$ is zero. Also, if the isotropy group under the action of G is trivial (or finite), then $\mu(x)$ must be zero.

Proof of Theorem. The necessity of (3) has been shown already. We shall show that it is sufficient.

As in [**4, 13**], we shall be working with generalized connections of class L_1^2, that is, connections which differ from a smooth connection by an element of the Sobolev space L_1^2. We shall also use gauge transformations in L_2^2. Since, as shown in [**1**], every L_2^2 orbit in the L_1^2 space of connections contains a C^∞ connection, there is no loss of generality as far as A is concerned. Also, since ϕ satisfies the elliptic equation $d_A''\phi=0$, we deduce by elliptic regularity that ϕ is C^∞. As explained in [**1**, §14], the group action and properties of curvature that we use extend without substantial change—in particular, $L_2^2\subset C^0$ so the topology of the line bundle is preserved.

We observe that the functional $\|\mu(A,\phi)\|_{L^2}^2$ on $\tilde{\mathcal{N}}$ is essentially the Yang–Mills–Higgs functional, Indeed, as shown above, if $(A,\phi)\in\mathcal{N}$,

$$\begin{aligned}\mathrm{YMH}_\tau(A,\phi) &= \left\|\Lambda F_A+\frac{i}{2}|\phi|_h^2-\frac{i}{2}\tau\right\|_{L^2}^2+2\pi\tau\deg(L) \\ &= \|\mu(A,\phi)\|_{L^2}^2+2\pi\tau\deg(L). \end{aligned} \tag{5}$$

The Yang–Mills–Higgs functional extends to a smooth functional for A and ϕ in the L_1^2 spaces. Notice that $\phi\in L^4$ since, as a particular case of the Sobolev inequalities, the inclusion $L_1^2\subset L^4$ is compact.

So given $D\in S^dX$, choose a smooth representative $(A_0,\phi_0)\in\tilde{\mathcal{N}}$, and consider the restriction of $\|\mu(A,\phi)\|_{L^2}^2$ to the orbit of (A_0,ϕ_0) under $(\tilde{\mathscr{G}}^{\mathbb{C}})^2$, the group of L_2^2 complex gauge transformations modulo $\mathbb{C}^*$. Take a minimizing sequence (A_n,ϕ_n) for $\|\mu\|_{L^2}^2$ in this orbit. Then for some constant C,

$$\|\mu(A_n,\phi_n)\|_{L^2}^2 < C.$$

This, together with equation (5), gives an L^2-bound on F_{A_n}. The main ingredient in the proofs of Donaldson and Hitchin referred to above is the weak compactness theorem of Uhlenbeck [19]. This states that if A_n is a sequence of L_1^2 connections for which F_{A_n} is bounded in L^2, then there are unitary gauge transformations u_n for which $u_n(A_n)$ has a weakly convergent subsequence. In our abelian situation, this is an easy consequence of the ellipticity of the Coulomb gauge. We have then a subsequence $A_{n'}$ and L_2^2 unitary gauge transformations $u_{n'}$ such that $u_{n'}(A_{n'})$ converges weakly in L_1^2. Rename $A_n = u_{n'}(A_{n'})$ and $\phi_n = u_{n'}(\phi_{n'})$. We shall now find L_1^2 uniform bounds for ϕ_n, so that by the weak compactness of L_1^2, the sequence ϕ_n will have a weakly convergent subsequence in L_1^2. To do this, consider the elliptic estimate

$$\|\phi_n\|_{L_1^2} \leqslant K_n(\|d''_{A_n}\phi_n\|_{L^2} + \|\phi_n\|_{L^2}).$$

We have $d''_{A_n}\phi_n = 0$; on the other hand, the constants K_n can be uniformly bounded since the d''_{A_n} converge. We need only find uniform bounds for $\|\phi_n\|_{L^2}$. First note that we have uniform bounds for $\|\phi_n\|_{L^4}$ as a consequence of (A_n, ϕ_n) being a minimizing sequence for $\|\mu(A,\phi)\|_{L^2}^2$, and the equality (5). Now Hölder's inequality

$$\|\phi_n\|_{L^2} \leqslant \operatorname{Vol}(X)^{1/4}\|\phi_n\|_{L^4}$$

gives us the uniform L^2-bounds for ϕ.

We conclude that (after possibly renaming again) (A_n, ϕ_n) converges weakly in L_1^2 to (A, ϕ). We need to show that (A, ϕ) is in the same orbit as (A_0, ϕ_0).

The (A_n, ϕ_n) are related to (A_0, ϕ_0) by elements $g_n \in (\mathscr{G}^{\mathbb{C}})^2$:

$$(A_n, \phi_n) = g_n \cdot (A_0, \phi_0)$$

satisfying

$$\|\phi_n\|_{L^2} = \|g_n\phi_0\|_{L^2} = \|\phi_0\|_{L^2}. \tag{6}$$

We shall prove first that g_n has a subsequence that converges to a holomorphic isomorphism g from (L, d''_{A_0}) to (L, d''_A). From this we may conclude that (A, ϕ) is in the same orbit as (A_0, ϕ_0) and is a solution to the vortex equations. The operators d''_{A_n} and d''_{A_0} are related by

$$d''_{A_n} - d''_{A_0} = \alpha_n,$$

where α_n is an element in the L_1^2 completion of the space of $(0,1)$-forms on X. The element α_n defines a class in $H^{0,1}(X)$, since by elliptic regularity the cohomology of the Dolbeault complex of L_1^2-forms is isomorphic to the cohomology of the ordinary Dolbeault complex. This class can be regarded as an element of $H^1(X, \mathcal{O})$ via the Dolbeault isomorphism. Consider the standard short exact sequence of sheaves defined by the exponential map

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 1,$$

and the corresponding long exact sequence in cohomology

$$0 \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O}^*).$$

Now, since $\alpha_n = g_n^{-1}d''g_n$, the image in $H^1(X, \mathcal{O}^*)$ of the class defined by α_n is the trivial element, thus this class comes from an element in $H^1(X, \mathbb{Z})$. The convergence of the α_n implies the convergence of the classes that they define, as can be seen by identifying these classes with their harmonic representatives. So, for n large enough, the integral class defined by α_n must be constant, thus by applying a fixed unitary gauge transformation we can assume that this class is zero. Hence there exists h_n such that $\alpha_n = d''h_n$.

The L_1^2 convergence of A_n implies the L_1^2 convergence of α_n. Now using the elliptic estimate

$$\|h_n - c_n\|_{L_2^2} \leqslant C\|d''h_n\|_{L_1^2},$$

where $c_n = \int_X h_n$ and C is a constant, we deduce that the L_1^2 convergence of $\alpha_n = d''h_n$ gives the L_2^2 convergence of $h_n - c_n$. To see this, recall that in general one has the estimate

$$\|f\|_{L_2^2} \leqslant C(\|d''f\|_{L_1^2} + \|f\|_{L^2}),$$

and the term $\|f\|_{L^2}$ can be omitted if f is L^2-orthogonal to the kernel of d''. But the kernel of d'' consists of the constant functions, and $f = h_n - c_n$ is certainly L^2-orthogonal to those. Since $L_2^2 \subset C^0$, we obtain a uniform bound on $h_n - c_n$, that is, there is a constant M such that

$$|h_n - c_n| < M. \tag{7}$$

Now,

$$g_n = K_n \exp(h_n - c_n) \tag{8}$$

for some non-zero constant K_n. From (7) and (8) we obtain

$$|K_n|e^{-M} \leqslant |g_n| \leqslant |K_n|e^M,$$

and then (6) gives

$$e^{-M} \leqslant |K_n| \leqslant e^M.$$

This shows that we can choose a subsequence of g_n which converges uniformly to a non-zero gauge transformation.

As mentioned above, the methods of our proof have already been used in dealing with other gauge-theoretical equations, like the anti-self-dual equations on a four-manifold, or Hitchin's self-duality equations on a Riemann surface [**13**]. The case of the vortex equations is, however, slightly different to the other cases. The difference is due to the fact that the relevant group for the vortex equations is U(n), while in the other cases it is typically SU(n). We have dealt here with the abelian case U(1), but it is conceivable that the same methods could work in the non-abelian situation studied in [**3**, **8**, **9**].

Our approach to solving the vortex equations is also useful in the study of the *moduli space of vortices*, that is, the space of all solutions modulo gauge equivalence. We have realized this space as the symplectic quotient

$$\{\mu(A, \phi) = 0\}/\tilde{\mathscr{G}}.$$

On the other hand, the existence theorem establishes a bijection

$$\{\mu(A, \phi) = 0\}/\tilde{\mathscr{G}} \overset{1-1}{\longleftrightarrow} \tilde{\mathscr{N}}/\tilde{\mathscr{G}}^{\mathbb{C}} \cong S^d X.$$

Under this identification, $S^d X$ inherits a symplectic structure which can be seen to be compatible with the complex structure. It defines then a Kähler structure whose study would be interesting to pursue.

ACKNOWLEDGEMENTS. It is a pleasure to thank my thesis supervisors, Professors Simon Donaldson and Nigel Hitchin, for many ideas and good advice. I also wish to thank Jørgen Andersen for discussions, and the referee for valuable comments. Thanks are due to the Spanish Consejo Superior de Investigaciones Científicas and

The British Council for financial support. Finally, I am very grateful to the IHES, where the final version of this paper has been written, for their hospitality and financial support.

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