Math 868 Fall 2018
Comments on homework by Alan Zhe Zhang

Homework 1 Comments

Scoring: Total 26 points

Problem	Points
1	3
2a,b	6 (3 each)
3	3
4a,b,c	3 (1 each)
5	2
6a,c,d,e, 7a,b, 8a,b,c	9 (1 for each part)

- Common mistakes

1. To get the estimate $|d(x_n, x_0) - d(x, x_0)| \leq d(x_n, x)$, one must apply the triangle inequality *twice:* In a metric space, if $a = d(x, y)$, $b = d(y, z)$ and $c = d(x, z)$, the triangle inequality gives $a \leq b + c$, and also $b \leq a + c$; together these give us $|a - b| \leq c$.

2. X is not open does not mean that X is closed. A simple example is the half-open interval $[0, 1)$ in $\mathbb{R}$.

5. One cannot prove that $B(x, D/2) \cap B(y, D/2) = \varnothing$ simply by drawing a picture. These are balls in an arbitrary metric space, however bizarre. A correct proof uses only the triangle inequality.

6. For part (e), $\beta(x) = 1 - h(|x|)$ is smooth, but not because of it is a composition of smooth functions ($|x|$ is not smooth). The key point is that $h(|x|)$ is a function of $|x|^2$, which is smooth.

7. For (b), it is not enough to draw a picture. Some also gave the following proof:

   - S^1 is open but $g^{-1}(S^1) = [0, 2\pi)$ is not open, so g^{-1} is not continuous.

   This is wrong for two reasons. First, the logic is wrong: one should take an open set in $[0, 2\pi)$, and check its preimage under $g^{-1} : S^1 \to [0, 2\pi)$. Second, the topology on the interval $I = [0, 2\pi)$ is the induced topology coming from the embedding $I \subset \mathbb{R}$ (in which the open sets in I are those of the form $U \cap I$ for U open in $\mathbb{R}$). In this topology, $[0, 2\pi)$ is in fact open.

   Here are two correct proofs of (7b):

   (a) Using sequences: Set $x_n = 2\pi - \frac{1}{n}$ and $y_n = g(x_n)$. Check that $\{y_n\}$ converges (to the point $1 \in S^1$), but $\{x_n\}$ does not. Thus g^{-1} is not continuous.

   (b) Note that $[0, \pi)$ is open in $[0, 2\pi)$ (why?). Its preimage $(g^{-1})^{-1}([0, \pi)) = [0, 2\pi)$ is the open upper half circle together with the point $1 \in \mathbb{C}$ and is not open (why?). Thus g^{-1} is not continuous.

Homework 2 Comments

Scoring: Total 32 points

Problem	Points
1	4
2	2
3	4 (2 each)
4	3
5	6 (3 each)
6	3
7	2
8	2 (1 each)
9	3
10	3

Common mistakes (by Problem number):

2. To write a vector or vector field X in matrix notation, write $X = \sum_i a^i(x) \frac{\partial}{\partial x^i}$ and form a vertical vector from the coefficients:

$$X = \begin{bmatrix} a^1(x) \\ a^2(x) \\ \vdots \\ . \end{bmatrix} \qquad \text{Do not form a "vector"} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \text{whose entries are basis vectors.}$$

3. Many of you used spherical coordinates to parameterize the surface. This is ok, but not necessary. One can simply start form the definition: For $v \in T_pS^2$, then locally there exists a curve

$$\gamma : (-\varepsilon, \varepsilon) \to S^2 \quad \text{with } \gamma(0) = p, \text{ and } \quad \frac{d(\phi \circ \gamma)}{dt}|_{t=0} = v.$$

Then, writing $\phi \circ \gamma = (a(t), b(t), c(t))$, we have $a^2(t) + b^2(t) + c^2(t) = 1$. Differentiating both sides, one gets $p \cdot v = 0$, this is the condition for a tangent vector in T_pS^2.

Notice that this method works for the unit sphere S^n of any dimension. I also encourage you to think about general hypersurfaces in $\mathbb{R}^n$.

4. Again, do not prove by drawing pictures, and start the proof from definitions is always a good idea.
In particular for problem 4, it is incorrect to assume that there is a t such that $\gamma([0,t]) \in X_1, \gamma([t,1]) \in X_2$ (for example, the path could go through X_1 and X_2 several times).

5.a) As in the hint, one has to use the fundamental theorem of calculus for a proof. The first step is to choose local coordinates; expressions like $\frac{\partial f}{\partial x^i}$ do not make sense only after local coordinates have been chosen.

After choosing local coordinates $\{x^i\}$ on a neighborhood of p in M and $\{y^j\}$ on a neighborhood of $f(p)$ in N, one can write f as $(f^1(x^1, \dots x^m), f^1(x^1, \dots x^m), \cdots)$ and use the fundamental theorem of calculus along the path $\gamma(t) = (1-t)p + tq, 0 \le t \le 1$ from p to q. This has the form

$$f^j(p) = f^j(q) + \int_0^1 \sum_i \frac{\partial f^j}{\partial x^i} \frac{d\gamma^i}{dt} \, dt. \tag{1}$$

This is equivalent to replacing f locally by the map $\tilde{f} = \psi f \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$.