Math 868 — Some Homework 9 Solutions

1. (a) Choosing bases for V and W gives isomorphisms $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, and then $L(V, W)$ is identified with the vector space of all $n \times m$ matrices, which has dimension mn.

   (b) Define a map $F : V^* \times W \to L(V, W)$ by letting $L = F(\alpha, w)$ be the linear map
$$L(v) = \alpha(v)\, w \qquad \text{for all } v \in V,\ \omega \in W,\ \alpha \in V^*.$$
   Then L is linear. Also, F is bilinear, so extends to a map $\overline{F} : V^* \otimes W \to L(V, W)$. This map $\overline{F}$ is injective because if L is the 0 linear transformation, then $\alpha(v) = 0$ for all $v \in V$, which means that $\alpha = 0$. But $V^* \otimes W$ and $L(V, W)$ both have dimension mn by part (a), so $\overline{F}$ is an isomorphism.

2. For each $p \in M$ and each non-zero $X \in T_pM$ we have
$$\tilde{g}(f_*X, f_*X) = (f^*\tilde{g})(X, X) = g(X, X) \geq 0$$
   because g is positive definite. This means that $f_*X \neq 0$ because $\tilde{g}$ is positive definite. Thus f_* is injective for each $p \in M$, which means that f is an immersion.

3. Choose coordinates with $p = (0, \ldots, 0)$ and $q = (d, 0, \ldots 0)$ where $d = dist(p, q)$. Then $\gamma_0(t) = (td, 0, \ldots, 0)$. For any path $\gamma : [0, 1] \to \mathbb{R}^n$ with $\gamma(0) = p$ and $\gamma(1) = q$, write $\gamma(t) = \gamma_0(t) + x(t)$ with $x(0) = x(1) = 0$. Then
$$|\dot{\gamma}(t)| \ = \ |\dot{\gamma}_0(t) + \dot{x}(t)| \ \geq \ |(\dot{\gamma}_0(t) + \dot{x}(t))^1| \ \geq \ |\dot{\gamma}_0(t)| + \dot{x}^1(t)$$
   since $\gamma_0^i(t) = 0$ for all t and all $i \neq 1$ and $\gamma_0^1(t) \geq 0$. Therefore
$$\begin{aligned} L_{g_0}(\gamma) \ &= \ \int_0^1 |\dot{\gamma}(t)|\, dt \ \geq \ \int_0^1 |\dot{\gamma}_0(t)| + \dot{x}^1(t)\, dt \\ &= \ L_{g_0}(\gamma_0) \ + \ \left(x^1(t)\right)\Big|_0^1 \\ &= \ L_{g_0}(\gamma_0) \ + 0 \qquad \square \end{aligned}$$

4. Given an orientation-preserving map $\phi(y^1, \ldots y^n) = (x^1, \ldots, x^n)$ we can write
$$\frac{\partial}{\partial y^i} \ = \ \sum_j \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \qquad \text{and} \qquad dy^i = \sum_j \frac{\partial y^i}{\partial x^i}\, dx^j.$$
   Let A be the matrix $\dfrac{\partial x^j}{\partial y^i}$, so A^{-1} is $\dfrac{\partial y^j}{\partial x^i}$. Then by the formula relating determinants and n-forms
$$dy^1 \wedge \cdots \wedge dy^n \ = \ A(dx^1) \wedge \cdots \wedge A(dy^n) \ = \ (\det A^{-1})\ dx^1 \wedge \cdots \wedge dx^n.$$
   Also, the matrix for the metric in terms of the y coordinates is
$$\tilde{g}_{ij} \ = \ g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right) \ = \ g\left(\sum A_i^k \frac{\partial}{\partial x^i}, \sum A_j^\ell \frac{\partial}{\partial x^\ell}\right) \ = \ \sum A_i^k A_j^\ell g_{k\ell}.$$
   Hence $\det \tilde{g} = (\det A)^2 \det g$ with $\det A > 0$ because ϕ preserves orientation.
   Altogether,
$$\begin{aligned} \sqrt{\det \tilde{g}}\ dy^1 \wedge \cdots \wedge dy^n \ &= \ \left(\det A \sqrt{\det g}\right)\ (\det A^{-1})\ dx^1 \wedge \cdots \wedge dx^n \\ &= \ \sqrt{\det g}\ dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$
   Thus the formula for the volume form is the same in any positively-oriented coordinate chart.