Math 868 — Homework 9

Due Friday Dec. 7.

1. (a) If V and W are vector spaces of finite positive dimensions n and m, prove that the set $L(V, W)$ of all linear maps $\lambda : V \to W$ is a vector space of dimension mn.

   (b) Prove that $V^* \otimes W \cong L(V, W)$.

2. Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds. Suppose $f : M \to \tilde{M}$ is a smooth map such that $f^*\tilde{g} = g$. Prove that f is an immersion.

3. Do Lee's Problem 13-10: Show that the shortest distance between two points in Euclidean space is a straight segment. More precisely, for $p, q \in \mathbb{R}^n$, let $\gamma_0 : [0, 1] \to \mathbb{R}^n$ be the path

$$\gamma_0(t) = (1 - t)p + tq.$$

   and let $\gamma : [0, 1] \to \mathbb{R}^n$ be another path with $\gamma(0) = p$ and $\gamma(1) = q$. Prove that

$$L_{g_0}(\gamma_0) \le L_{g_0}(\gamma)$$

   where g_0 is the standard metric on $\mathbb{R}^n$, given by the dot product. *Hint:* choose coordinates with origin at p and with $q = (d, 0, \ldots, 0)$, write $\gamma(t) = \gamma_0(t) + x(t)$ and note that $|\dot{\gamma}_0 + \dot{x}| \ge \dot{\gamma}_0 + \dot{x}^1$.

4. Let (M, g) be an oriented Riemannian manifold. As in class, the metric determines a volume form that is given in each positively-oriented coordinate chart by

$$dvol_g = \sqrt{\det g_{ij}}\ dx^1 \wedge \cdots \wedge dx^n.$$

   Show directly that this does not depend on the choice of the positively-oriented chart, that is, if $\phi(y^1, \ldots y^n) = (x^1, \ldots, x^n)$ is a diffeomorphism between charts with $\det D\phi > 0$, then $\phi^* dvol_g$ has the same expression in y-coordinates.

   Hint: Write $\frac{\partial}{\partial y^i} = \sum_j \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$ and $dy^i = \sum_j \frac{\partial y^i}{\partial x^i}\, dx^j$, and let $A^i_j = \frac{\partial x^j}{\partial y^i}$ be the matrix of $D\phi$ in these coordinates. Use the properties of determinants and the formula related determinants to wedge products.

5. On a Riemannian manifold (M, g), each function $f \in C^\infty(M)$ determines a *gradient vector field* ∇f (which depends on g) by the equation

$$g(\nabla f, X) \ = \ df(X) \qquad \text{for all vector fields } X$$

   (see Lee, bottom of page 342). Note that $df(X)$ can also be written as Xf.

   Fix f and a regular point p of f and do Lee's Problem 13-21:

   (a) Show that, for unit vectors $X \in T_pM$, the directional derivative $(Xf)_p$ is maximal when X points in the direction of $(\nabla f)_p$.

   Hint: Find a statement of the Cauchy-Schwarz inequality that includes a precise statement of when equality holds.

   (b) Show that $|\nabla f|$ is equal to the value of the directional derivative in that direction.

   (c) Show that $(\nabla f)_p$ is normal to the level set of f through p.

   (d) Finally, show that in local coordinates ∇f is $\sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \ \frac{\partial}{\partial x^j}$.

6. Do Lee's Problem 16-19: On $\mathbb{R}^n$ with the euclidean metric and the standard orientation form $dx^1 \wedge \cdots \wedge dx^n$,

   (a) Calculate $*dx^i$ for $i = 1, \ldots, n$.

   (b) For $n = 4$, calculate $*(dx^i \wedge dx^j)$ for $(i, j) = (1, 2),\ (1, 3),\ (1, 4),\ (2, 3),\ (2, 4)$, and $(3, 4)$.

7. Do the following version of Lee's Problem 16-20: Let M be an oriented Riemannian manifold. A 2-form ω on M is called *self-dual* if $*\omega = \omega$, and *anti-self-dual* if $*\omega = -\omega$.

   (a) Show that every 2-form ω can be uniquely written as the sum of a self-dual 2-form α and an anti-self-dual 2-form β.

   (b) On $M = \mathbb{R}^4$ with the standard metric and orientation, write down a basis of the space of constant self-dual 2-forms, and a similar basis for the anti-self-dual forms (both are 3-dimensional vector spaces). *Use your answer to Problem 5(b) above.*

   (c) What is the general form of a self-dual 2-form ω on $\mathbb{R}^4$ in standard coordinates $\{x^i\}$?
   Its coefficients are three functions $f, g, h \in C^\infty(R^4)$.

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2. For each $p \in M$ and each non-zero $X \in T_pM$ we have
$$\tilde{g}(f_*X, f_*X) = (f^*\tilde{g})(X,X) = g(X,X) \geq 0$$
because g is positive definite. This means that $f_*X \neq 0$ because $\tilde{g}$ is positive definite. Thus f_* is injective for each $p \in M$, which means that f is an immersion.

3. Choose coordinates with $p = (0,\ldots,0)$ and $q = (d,0,\ldots 0)$ where $d = dist(p,q)$. Then $\gamma_0(t) = (td,0,\ldots,0)$. For any path $\gamma : [0,1] \to \mathbb{R}^n$ with $\gamma(0) = p$ and $\gamma(1) = q$, write $\gamma(t) = \gamma_0(t) + x(t)$ with $x(0) = x(1) = 0$. Then
$$|\dot\gamma(t)| \;=\; |\dot\gamma_0(t) + \dot x(t)| \;\geq\; |(\dot\gamma_0(t) + \dot x(t))^1| \;\geq\; |\dot\gamma_0(t)| + \dot x^1(t)$$
since $\gamma_0^i(t) = 0$ for all t and all $i \neq 1$ and $\gamma_0^1(t) \geq 0$. Therefore
$$\begin{aligned} L_{g_0}(\gamma) \;&=\; \int_0^1 |\dot\gamma(t)|\; dt \;\;\geq\;\; \int_0^1 |\dot\gamma_0(t)| + \dot x^1(t)\; dt \\ &=\; L_{g_0}(\gamma_0) \;+\; \big(x^1(t)\big)\Big|_0^1 \\ &=\; L_{g_0}(\gamma_0) \;+0 \qquad\qquad \square \end{aligned}$$

4. Given an orientation-preserving map $\phi(y^1,\ldots y^n) = (x^1,\ldots,x^n)$ we can write
$$\frac{\partial}{\partial y^i} \;=\; \sum_j \frac{\partial x^j}{\partial y^i}\,\frac{\partial}{\partial x^j} \qquad \text{and} \qquad dy^i = \sum_j \frac{\partial y^i}{\partial x^i}\; dx^j.$$
Let A be the matrix $\dfrac{\partial x^j}{\partial y^i}$, so A^{-1} is $\dfrac{\partial y^j}{\partial x^i}$. Then by the formula relating determinants and n-forms
$$dy^1 \wedge \cdots \wedge dy^n \;=\; A(dx^1) \wedge \cdots \wedge A(dy^n) \;=\; (\det A^{-1})\; dx^1 \wedge \cdots \wedge dx^n.$$
Also, the matrix for the metric in terms of the y coordinates is
$$\tilde g_{ij} \;=\; g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^i}\right) \;=\; g\left(\sum A_i^k \frac{\partial}{\partial x^i},\; \sum A_j^\ell \frac{\partial}{\partial x^\ell}\right) \;=\; \sum A_i^k A_j^\ell g_{k\ell}.$$
Hence $\det \tilde g = (\det A)^2 \det g$ with $\det A > 0$ because ϕ preserves orientation.
Altogether,
$$\begin{aligned} \sqrt{\det \tilde g}\; dy^1 \wedge \cdots \wedge dy^n \;&=\; \left(\det A \sqrt{\det g}\right)\; (\det A^{-1})\; dx^1 \wedge \cdots \wedge dx^n \\ &=\; \sqrt{\det g}\; dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$
Thus the formula for the volume form is the same in any positively-oriented coordinate chart.

5. Using the geodesic equation $\ddot x^k = -\Gamma^k_{pq}\dot x^p \dot x^q$ and using summation convention, we have
$$\tfrac12 |\dot\gamma(t)|^2 \;=\; \frac{d}{dt}\; g_{ij}\dot x^i \dot x^j \;=\; \partial_k g_{ij}\dot x^k \dot x^i \dot x^j + 2g_{ki}\dot x^i \ddot x^k \;=\; \partial_k g_{ij}\dot x^k \dot x^i \dot x^j - 2g_{ki}\Gamma^k_{pq}\dot x^p \dot x^q \dot x^i.$$
Changing the index k in the last term to ℓ and using the definition of the Christoffel symbols, this becomes
$$\begin{aligned} \tfrac12 |\dot\gamma(t)|^2 \;&=\; \left[\partial_k g_{ij} - 2g_{\ell i}\cdot \tfrac12 g^{\ell m}\left(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}\right)\right]\, \dot x^i \dot x^j \dot x^k \\ &=\; \left[\partial_k g_{ij} - \left(\partial_j g_{ki} + \partial_k g_{ji} - \partial_i g_{jk}\right)\right]\, \dot x^i \dot x^j \dot x^k \end{aligned}$$
and this vanishes because $\dot x^i \dot x^j \dot x^k$ is symmetric in i,j,k.