Math 868 — Homework 8

Due Monday, Nov. 26

These problems are applications of Stokes' Theorem and the definition of DeRham cohomology.

1. Let H be the upper hemisphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1, z \geq 0\}$. Evaluate

$$\int_{\partial H} (x + y)\, dz + (y + z)\, dx + (x + z)\, dy$$

directly and by Stokes' Theorem. (Use orientation form $dx \wedge dy$.)

2. Let R be a region in R^3 oriented by $dx \wedge dy \wedge dz$, and let $\omega = \frac{1}{3}\Big(z\, dx \wedge dy + x\, dy \wedge dz + y\, dz \wedge dx\Big)$.

   (a) Show that $\int_{\partial R} \omega = \mathrm{Vol}(R)$.

   (b) Show that $d(\omega/r^3) = 0$, where $r^2 = x^2 + y^2 + z^2$

   (c) Deduce that $H^2(S^2) \neq 0$, i.e. that there is a closed 2-form on S^2 that is not exact.

3. Suppose that a manifold M is the disjoint union of two components M_1 and M_2. Explain why its DeRham cohomology is $H^*(M) = H^*(M_1) \oplus H^*(M_2)$.

4. Let M be an oriented n-manifold, and X is a compact, oriented p-dimensional submanifold of M. Define a map

$$I_X : \Omega^p(M) \to \mathbb{R} \qquad \text{by} \qquad I_X(\omega) \;=\; \int_X \omega.$$

   (a) Verify that I_X is linear.

   (b) Show that if $\omega_1, \omega_2 \in \Omega^p(M)$ are cohomologous then $I_X(\omega_1) = I_X(\omega_2)$. Consequently, I_X induces a linear map

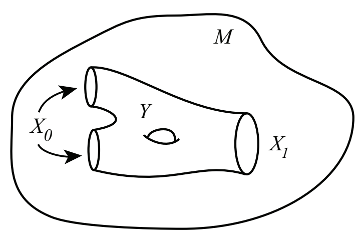
$$\overline{I}_X : H^p(M) \to \mathbb{R}.$$

   (c) Suppose that $X = \partial Y$ is the boundary of a compact, oriented $(p+1)$-dimensional submanifold $Y \subset M$. Show that $\overline{I}_X \equiv 0$.

Definition. Two compact oriented submanifolds X_0 and X_1 of M are *cobordant in M* if there is a compact oriented submanifold $Y \subset M$ with ∂Y is the disjoint union

$$\partial Y \;=\; X_1 \cup (-X_0),$$

where $-X_0$ denotes the manifold X_0 with its orientation reversed.



   (d) Show that if X_0 and X_1 are cobordant then the linear functionals $\overline{I}_{X_0}$ and $\overline{I}_{X_1}$ are equal.

5. Read the statement and proof of the "Zigzag Lemma" on page 461-2 of Lee (also done in class). Lee ends with three assertions:

   (a) The cohomology class $[a]$ is independent of the choices made,

   (b) δ is linear, and

   (c) The resulting long exact sequence is exact.

Of these, (a) was done is class and (b) is clear. Your task: verify (c). Note that this requires showing three inclusions $\ker d \subset \operatorname{im} d$ and three $\operatorname{im} d \subset \ker d$.

6. Let $X = S^n \setminus A$ where A is the union of $k \geq 1$ disjoint disks D_i. Use the Mayer-Vietoris sequence to compute the DeRham cohomology $H^*(X)$. *Hint:* begin by noting that $S^n \setminus D_1$ is diffeomorphic to R^n.