Math 868 — Some Solutions to Homework 7

(2) (a) Because M_1 and M_2 are orientable, we can find nowhere-vanishing forms $\sigma_1 \in \Omega^n(M_1)$ and $\sigma_2 \in \Omega^n(M_2)$. Because $\Lambda^n T^*M$ is one-dimensional, there is a function f such that $\sigma_1 = f\sigma_2$ on $M_1 \cap M_2$. Furthermore, since σ_1 and σ_2 vanish nowhere and $M_1 \cap M_2$ is connected, we have either $f > 0$ or $f < 0$ on $M_1 \cap M_2$. After replacing σ_2 by $-\sigma_2$ if necessary, we can assume that $f > 0$. Let $\{\beta_1, \beta_2\}$ be a partition of unity subordinate to the cover $\{M_1, M_2\}$ of M. Then

$$\sigma = \beta_1\sigma_1 + \beta_2\sigma_2$$

is an n-form on M that vanishes nowhere because $\sigma = \sigma_1$ on $M_1 \setminus M_2$, $\sigma = \sigma_2$ on $M_2 \setminus M_1$, and $\sigma = (\beta_1 + f\beta_2)\sigma_1$ with $\beta_1 + f\beta_2 > 0$ on $M_1 \cap M_2$. Hence σ is an orientation form for M, so M is orientable.

(b) Write S^n as $M_1 \cup M_2$ where M_1 (resp. M_2) is an neighborhood of the northern (resp. southern) hemisphere, each diffeomorphic to the n-disk, so that $M_1 \cap M_2$ is a connected neighborhood of the equation. Then M_1 and M_2 are orientable and hence S^n is connected by part (a).

(3) M and N are diffeomorphic, so have the same dimension n. Because they are orientable we can fix orientation forms σ_1 on M and σ_2 on N; both are nowhere-vanishing n-forms. As in Problem 2, there is a function f on M such that $\phi^*\sigma_2 = f \cdot \sigma_1$. If we show that f vanishes nowhere then, since M is connected, we have either $f > 0$ (which means that ϕ is orientation-preserving), or $f < 0$ (which means that ϕ is orientation-reversing).

Thus it suffices to fix $p \in M$ and show that $f(p) \neq 0$. Since ϕ is a diffeomorphism, the Local Immersion Theorem implies that there are coordinates $\{x^i\}$ around p and $\{y^i\}$ around $\phi(p)$ such that $\phi(x^1, \ldots, x^n) = (y^1, \ldots, y^n)$. In these coordinates $\sigma_1 = \lambda\, dx^1 \wedge \cdots \wedge dx^n$ and $\sigma_2 = \mu\, dy^1 \wedge \cdots \wedge dy^n$ for non-vanishing functions λ and μ. Then

$$\begin{aligned} \phi^*\sigma_2 &= (\mu \circ \phi)\,(\phi^* dy^1) \wedge \cdots \wedge (\phi^* dy^n) \\ &= (\mu \circ \phi)\,(dx^1 \wedge \cdots \wedge dx^n) \\ &= \lambda^{-1}(\mu \circ \phi)\; \sigma_1 \end{aligned}$$

where $f = \lambda^{-1}(\mu \circ \phi)$ vanishes nowhere. $\square$

(5) Let $\alpha : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map $\alpha(x) = -x$; this restricts to the antipodal map $\alpha : S^n \to S^n$. As in Proposition 13.14 in Lee, the orientation form of S^n is the restriction of

$$\sigma = \iota_N(dx^1 \wedge \cdots \wedge dx^{n+1})$$

to S^n, where N is the outward unit normal $N(x) = x$ for $x \in S^n$. Note that $\alpha^{-1} = \alpha$ and that $\alpha_*(N(x)) = -x = N(-x)$. Therefore

$$\begin{aligned} \alpha^*\sigma &= \iota_{(\alpha^{-1})_* N}\; \alpha^*(dx^1 \wedge \cdots \wedge dx^{n+1}) \\ &= \iota_N(\alpha^* dx^1 \wedge \cdots \wedge \alpha^* dx^{n+1}) \\ &= (-1)^{n+1}\iota_N(dx^1 \wedge \cdots \wedge dx^{n+1}) \\ &= (-1)^{n+1}\sigma. \end{aligned}$$

Thus the antipodal map is orientation-preserving if and only if n is odd.