Math 868 — Homework 7

Due Wednesday, Nov. 14

Problems on Orientations. Problems 2-4 below are re-wordings of Lee's Problems 15-1, 15-2, and 15-3.

1. Use orientation forms to show that:
   (a) An open subset $U \subset M$ of an orientable manifold is orientable.
   (b) The product $M \times N$ of two orientable manifolds is orientable.

2. Suppose that a manifold M is the union of two oriented open submanifolds M_1, M_2 whose intersection $M_1 \cap M_2$ is connected.
   (a) Prove that M is orientable.
   Hint: We know there exists a function β with support in M_1, with $\beta(x) \geq 0$ for all x, and with $\beta \equiv 0$ on $M \setminus M_2$.
   (b) Use this to give a proof that S^n is orientable.

3. Let $\phi : M \to N$ be a smooth map between manifolds which is a local diffeomorphism (so DF is invertible). Show that if M is connected then ϕ is either orientation-preserving, or orientation-reversing (see the definitions in Lee on page 383).

4. For $n \geq 1$, let S^n be the unit sphere in $\mathbb{R}^{n+1}$, and let $\alpha : S^n \to S^n$ be the antipodal map, defined by $\alpha(x) = -x$. Show that α is orientation preserving if and only if n is odd. *Hint:* By Problem 2 above, it suffices to compare orientations for $(D\alpha)_p : T_pS^n \to T_{-p}S^M$ where p is the north pole.

Problems on Integration. Read Lee, pages 400–410 for background.

5. Evaluate $\int_S x\,dy \wedge dz + y\,dx \wedge dy$ where S is the oriented surface parameterized by
$$\phi : [0,1] \times [0,1] \to \mathbb{R}^3$$
by $\phi(u,v) = (u+v, u^2 - v^2, uv)$ and oriented by the orientation form $dx \wedge dy$.

6. Let $A = \{(x,y,z) \,|\, y = x^2 + z^2, y \leq 4\}$, oriented by the orientation form $dz \wedge dx$. Evaluate:
$$(a) \quad \int_A z\,dx \wedge dy \qquad \text{and} \qquad (b) \quad \int_A e^y \; dz \wedge dx.$$
Hint: use polar coordinates in the (x,z)-plane.

7. Do Problem 16-2 on page 434 of Lee.