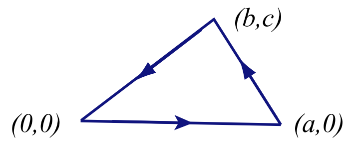
Math 868 — Homework 5

Due Wednesday, Oct. 24

1. Follow Lee page 274, to answer this: Let V be a vector space with dual space V^*. Assume that V has a countable basis.
   (a) Define a linear map $\phi : V \to V^{**}$ that is natural (ie defined without using any basis).
   (b) Prove that ϕ is injective.
   (c) When V is finite-dimensional, prove that ϕ is an isomorphism.

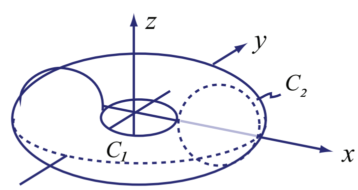
2. Let $c : [0,1] \to \mathbb{R}^3$ be the path $c(t) = (t, t^2, t^3)$. Evaluate $\int_C \omega$ where $\omega = y\ dx + 2x\ dy + y\ dz$.

3. For $\alpha = \frac{1}{2}(x\ dy - y\ dx)$, find
   (a) $\int_{C_R} \alpha$ where C_R is the circle $\{x^2 + y^2 = R^2\}$ in $\mathbb{R}^2$ with counterclockwise orientation.
   (b) $\int_T \alpha$ where T is the oriented triangle in $\mathbb{R}^2$ shown.



4. Determine whether each form is exact. If it is, find all functions f such that $\omega = df$.
   (a) $\omega = xy\,dx + \frac{1}{2}x^2\,dy$ on $\mathbb{R}^2$.
   (b) $\omega = x\,dx + xz\,dy + xy\,dz$ on $\mathbb{R}^3$.
   (c) $\omega = y\,dx$ on $\mathbb{R}^2$.
   (d) $\omega = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)(y\ dx - x\ dy)$ on $\{(x,y) \,|\, x \neq 0 \text{ and } y \neq 0\}$.

5. Let ω be the 1-form $\omega = \left(\frac{-y}{x^2+y^2}\right)\ dx + \left(\frac{x}{x^2+y^2}\right)\ dy$ on $\mathbb{R}^3 - \{z - \text{axis}\}$. Find $\int_{C_1} \omega$ and $\int_{C_2} \omega$ where C_1 and C_2 are the two curves shown on the torus of radius 1 around the core circle of radius 2 in the xy plane.



6. Write down a 1-form η on $\left\{(x,y,z) \,\middle|\, z \neq 0 \text{ and } x^2 + y^2 \neq 4\right\}$ so that the numbers $\int_{C_1} \eta$ and $\int_{C_1} \eta$ are the same as integrals of ω in Problem 5 but in the opposite order. Verify by integrating.

7. Let V be an n-dimensional vector space, and $\omega \in \Lambda^2(V^*)$.
   (a) Show that there is basis $\{e^1, e^2, \ldots e^n\}$ of V^* such that
   $$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2r-1} \wedge e^{2r} \qquad \text{for some } r.$$
   (b) Show that $\omega^r \neq 0$ but $\omega^{r+1} = 0$. Thus r, called the *rank* of ω, depends only on ω.

Some solutions and hints to the above problems:

2. $\frac{34}{15}$.
3. In both (a) and (b) the integral is equal to the area enclosed by the path.
4. (a) and (d) are exact. Once you find f, you can check yourself that it works.
5. Convert to cylindrical coordinates (r, θ, z), then do the integrals.
6. Ask: what should be the singular set? Then use coordinates (r, θ, z) again.
7. Steps:
   (a) Identify V with $\mathbb{R}^n$. Define a skew-symmetric matrix A by $A_{ij} = \omega(e_i, e_j)$ where $\{e_i\}$ is the standard basis.
   (b) If $\omega \neq 0$ then there is a v such that $Av \neq 0$. Use the fact that the dot product in $\mathbb{R}^n$ satisfies $Ax \cdot y = x \cdot A^T y$ to show that v and Av are linearly independent.
   (c) Set $f_1 = v,\ f_2 = Av$, and complete these to a basis $\{f_1, f_2, f_3, \dots\}$ of V. Show that, in this basis, $\omega = \sum A_{ij}\, f^i \wedge f^j$ with $A_{12} \neq 0$.
   (d) Then set
   $$\begin{cases} v_1 = f_1 - \frac{1}{A_{12}} \sum_{\ell \geq 3} A_{2\ell}\, f^\ell \\ v_2 = A_{12} f^2 + A_{13} f^3 + \cdots + A_{1n} f^n. \end{cases}$$
   (e) Proceed by induction.