Some HW4 solutions

4. Identify $Mat_n = \{n \times n \text{ real matrices}\}$ with $\mathbb{R}^{n^2}$ and define $\Phi : Mat_n \to Mat_n$ by $\phi(A_{ij}) = (AA^T)_{ij} = \sum_j A_{ij}A_{ij}$. Then Φ is continuous (it is a quadratic polynomial!), so $O(n) = \Phi^{-1}(Id.)$ is closed. Furthermore, for each $A \in O(n)$, we have $\sum_j A_{ij}^2 = 1$, so $\|A\|^2 = \sum_{ij} A_{ij}^2 = n$. Thus $O(n)$ is closed and bounded in $\mathbb{R}^{n^2}$, so compact.

5. If $A \in T_I O(n)$ there is a path $B(t)$ in $O(n)$ with $B(0) = I$ and $\dot{B}(0) = A$ satisfies $B^T(t)B(t) = I$. Applying $\frac{d}{dt}$ and evaluating at $t = 0$ gives $0 = \dot{B}^T(0) \cdot I + I \cdot \dot{B}^T(0) = A^T + A$, so A is skew-symmetric. Thus $T_I O(n) \subset \mathfrak{o}(n) = \{n \times n \text{ skew-symmetric matrices}\}$. Conversely, if $A \in \mathfrak{o}(n)$, then $B(t) = e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \cdots$ satisfies $B(0) = I$, $\dot{B}(0) = A$ and
$$B^T(t)B(t) \;=\; e^{tA^T}e^{tA} \;=\; e^{-tA}e^{tA} \;=\; I.$$
Consequently, $B(t)$ lies in $O(n)$ for all t and hence $A \in T_I O(n)$. Therefore $T_I O(n) = \mathfrak{o}(n)$.

6. (a) First, $U(n)$ is a group: if $A, B \in U(n)$ then (i) $(AB)^*(AB) = B^*A^*AB = B^*B = I$ and (ii)$A^*A = I \implies A^{-1} = A^* \implies (A^{-1})^* = A^{**} = A \implies (A^{-1})^*A^{-1} = I$. Thus $AB \in U(n)$ and $A^{-1} \in U(n)$.

   Let $H(n) = \{n \times n \text{ cx. matrices} \,|\, A^* = A\}$ be the vector space of hermitian matrices and define $\Phi : \mathbb{C}^{n^2} \to H(n)$ by $\Phi(A) = A^*A$. This is smooth (it is quadratic in the entries of A), $\Phi^{-1}(Id) = U(n)$, and the image lies in $H(n)$ since $(A^*A)^* = A^*A^{**} = A^*A$. One shows
$$d\Phi_A \text{ is onto}$$
   exactly as was done in class for $O(n)$ with A^T replaced everywhere by A^*. Hence $U(n)$ is an immersed submanifold of $\mathbb{C}^{n^2}$. Finally, the group operations are smooth because they are restrictions of the smooth group operations of $GL(n, \mathbb{C})$ to the submanifold $U(n)$.

   (b) Repeating Problem 5 above, with A^T replaced everywhere by A^*, shows that $T_I U(n)$ is the space $\mathfrak{u}(n) = \{n \times n \text{ cx. matrices} \,|\, A^* = -A\}$ of skew-hermitian matrices.

   Alternatively, one can show $T_I U(n) \subset \mathfrak{u}(n)$ and then use a dimension count. For this, note that if $A = B + Ci$ then $A^* = A$ iff B is symmetric and C is skew-symmetric. Hence
$$\dim H(n) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2 \quad \text{and similarly} \quad \dim \mathfrak{u}(n) = n^2.$$

7. (a) For any $x, g, h \in G$ we have $L_gL_h(x) = L_g(hx) = ghx = L_{gh}(x)$. In particular, $L_gL_{g^{-1}}(x) = x$ and $L_{g^{-1}}L_g(x) = x$. Thus $L_gL_h = L_{gh}$ and L_g is a diffeomorphism with inverse $L_{g^{-1}}$.

   (b) By assumption there is a neighborhood U of $I \in G$ and a chart $\phi : U \to V \subset \mathbb{R}^n$ on which the group operations are smooth. For each $g \in G$ set $U_g = L_g(U)$ and let $\phi_g : U_g \to V$ be $\phi_g = \phi \circ L_{g^{-1}}$. Then U_g is a neighborhood of g and ϕ_g is a bijection because it is the composition of two bijection). Define an atlas by
$$\mathcal{A} \;=\; \{(U_g, \phi_g) \,|\, g \in G\}.$$
   These U_g cover G. We will show that whenever $U_g \cap U_h \neq \emptyset$ the transition map $\phi_h^{-1}\phi_g : U_g \cap U_h \to U_g \cap U_h$ is smooth. For this, first note that
$$\phi_h\phi_g^{-1} \;=\; \phi \circ L_{h^{-1}}\,(\phi \circ L_{g^{-1}})^{-1} \;=\; \phi \circ L_{h^{-1}} \circ L_g \circ \phi^{-1} \;=\; \phi \circ L_{h^{-1}g} \circ \phi^{-1}.$$

This looks smooth, but be careful: *we only know that L_g is smooth* for $g \in U$. To deal with this problem, fix $x \in U_g \cap U_h$. Since $x \in U_h = L_h U$ we have $h^{-1}x \in U$, so $L_{h^{-1}x}$ is smooth by the hypothesis. Similarly, since $x \in U_g$ we have $g^{-1}x \in U$, so $L_{g^{-1}x}$ is smooth and hence so is its inverse. Therefore

$$L_{h^{-1}g} = L_{(h^{-1}x)(x^{-1}g)} = L_{h^{-1}x} \circ \left(L_{g^{-1}x}\right)^{-1}$$

is smooth. This shows that all transition maps are smooth, so $\mathcal{A}$ defines a differentiable structure on G.

(c) One should also note that the assumption that G is a topologicial Lie group means that it is a topological manifold and hence, with out definitions, it is a metric space.

Homework 4 Comments

Scoring: Total 23 points

Problem	Points
1	4
2	3
3	2
4	2
5	3
6	4 (2 each)
7	2+(3 bonus points)

Common mistakes (by Problem number):

1. The basic idea is to choose a path $\sigma : [0, 1] \to M$ from p to q, and extend the tangent vector field along the path to a vector field on all of M. One must use a cutoff function (or a partition of unity) to make the extended vector field *compactly supported*. The flow that it generates is then complete (by Theorem 9.16 Lee).

   A complete flow is needed to ensure that Φ_t is defined for all t, and that its domain is all of M; in particular, is a diffeomorphism for $t = 1$. For vector fields that are not compactly supported, one knows only that the diffeomorphism is globally defined only in a small neighborhood of $t = 0$.

2(b). One common mistake is that the target space should be fixed when defining a homotopy. In particular, for maps $S^1 \to S^n$, each intermediate map $f_t = F(t, \cdot)$ must be a map *into* S^n.

Another mistake is that in order to use stereographic projection from the north pole N, one need to ensure $N \notin f(S^1)$. This can be done by a rotation, provided that there is *some* point $p \notin f(S^1)$. This is basically obvious, and I gave full credit if you just assumed it.

Here is a proof assuming that f is C^1: If $(df)_x = 0$ for all $x \in S^1$, then f is a constant map, so of course there is a $p \notin f(S^1)$. Otherwise, fix x with $(df)_x \neq 0$. By the local immersion theorem there is a neighborhood $U = (x - \varepsilon, x + \varepsilon)$ of x and coordinates around p such that $f(t) = (t, 0, \ldots, 0)$ for $t \in U$.

Remark. Even assuming that f is only continuous, one can first homotope f to a nonsurjective map, see Proposition 1.14 in A. Hatcher's book for a proof. The basic idea: for $x \in S^2$, let $B \subset S^2$ be a small open ball centered at x. Then $f^{-1}(B)$ is a union of intervals in $(0, 1)$ and $f^{-1}(x)$ is finite, so there are finitely many intervals in $f^{-1}(B)$ containing x. For each of these intervals push the image to the boundary of B. In this way we homotope f to a new map which avoid x in its image.

4 . $O(n)$ is closed simply because $O(n) = F^{-1}(I)$, where $F(A) = AA^T$ continuous, and I as a single point is compact. The Regular Preimage Theorem is not needed.

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