Math 868 — Homework 4

Due Friday, Sept. 28

1. (Lee Problem 9.7). Let M be a connected ($\Rightarrow$ path connected) manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. Do this in two steps:

   (a) Prove this for the case $M = B^n$ (the open unit ball in $\mathbb{R}^n$). *Hint*: show that there is a compactly supported vector field X on B^n whose flow ϕ satisfies $\phi_1(p) = q$.

   (b) Prove the general case by repeatedly applying (a).

   Three definitions:

   - An *isotopy* of a manifold M is a smooth map $\Phi : [0, 1] \times M \to M$ such that $\Phi_t : M \to M$ is a diffeomorphism for all $t \in [0, 1]$.
   - Two embedded submanifolds $f : S \to M$ and $g : S \to M$ are *isotopic* if there is an isotopy Φ of M with $\Phi_0 = Id.$ and $\Phi_1 \circ f = g$.
   - A weaker notion: Two maps $f, g : S \to M$ are *homotopic* if there is a map $F : [0, 1] \times S \to M$ such that, writing $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in S$ (i.e. $f_t = F(t, \cdot)$ interpolates between f and g).

2. (a) Let $f : S^2 \to \mathbb{R}^3$ be the embedding of the unit sphere, and let E be the ellipse

   $$E \;=\; \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, 4x^2 + 9y^2 + z^2 = 1\right\}.$$

   Parameterize E by a map $g : S^2 \to \mathbb{R}^3$, and show that g is isotopic to f.

   (b) Now consider $M = S^n$. Prove that any map $f : S^1 \to S^n$ is **homotopic** to a constant map $g : S^1 \to S^n$ (i.e. a map whose image is a single point).

3. (Lee Problem 9.16). Give an example of smooth vector fields X, Y, Z on $\mathbb{R}^2$ such that $X = Y = \frac{\partial}{\partial x}$ along the x-axis, but $\mathcal{L}_X Z \neq \mathcal{L}_Y Z$ at the origin. (This shows that $\mathcal{L}_X Z$ at a point p depends on the derivatives of X at p, not just the vector X_p.)

4. Prove that the orthogonal group $O(n)$ is compact. (*Hint:* first show that if $A = (A_{ij})$ is orthogonal then $\sum_j A_{ij}^2 = 1$ for each i).

5. Verify that the tangent space to $O(n)$ at the identity matrix I is the vector space of all skew-symmetric $n \times n$ matrices, that is, the matrices A with $A^t = -A$. (Consider paths $B(t) = I + tA + \cdots$ in $O(n)$.)

6. The unitary group $U(n)$ is the group of all $n \times n$ complex matrices A such that $A^*A = Id.$, where $A^* = \overline{A}^t$ is the conjugate of the transpose (= transpose of the conjugate).

   (a) Prove that $U(n)$ is a Lie group (adapt the proof given in class for $O(n)$).

   (b) What is the tangent space to $U(n)$ at the identity matrix?

7. If G is a Lie group then, for each $g \in G$, left multiplication by g is a map $L_g : G \to G$ by $L_g h = gh$; this is smooth by the definition of Lie group.

   (a) Show that $L_g L_h = L_{gh}$ and that L_g is a diffeomorphism.

   (b) Prove that a topological Lie group G that is locally a smooth Lie group near $Id. \in G$ is a manifold. [1]

   Hint: if $\phi : U \to \mathbb{R}^n$ is a chart at $Id.$, then $\phi \circ L_{g^{-1}}$ is a chart at g. Show that the transition maps are diffeomorphisms, so define an atlas.

---

[1] A *topological Lie group* is a group that is a topological manifold such that the group operations $g \mapsto g^{-1}$ and $(g, h) \to hg$ are continuous. *Locally a smooth Lie group near* $Id.$ means there is a coordinate neighborhood U of $Id.$ in which these operations are smooth.