# Homework 3 Solutions

**Scoring:** Total 23 points

| Problem | Points |
|---|---|
| 1 | 4 |
| 2 | 4 + up to 4 points bonus |
| 3 | 5 |
| 4 | 2 |
| 5 | 3 |
| 6 | 5 |

**Common mistakes by Problem number:**

1. Correction definitions:

   (a) Given any compact subset $K$ of $X$, exist $N(K) \in \mathbb{N}^+$, such that for any $n \geq N$, $x_n \notin K$.

   (b) $\#\{x_n\} \cap K < \infty$, for any compact subset $K$ of $X$.

   For definition one, notice that $N$ depends on K, so it is incorrect to define as:
   There exists a $N \in \mathbb{N}^+$, such that for any $n \geq N$, and any compact subset $K$ of $X$, $x_n \notin K$.

4&6. In order to apply regular preimage theorem, we need to define an 'appropriate' map in the sense that it is smooth, and has the desired level set as given in the problems.
For problem 4, define as:

$$\begin{aligned} F : M \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto y - f(x) \end{aligned}$$

For problem 6, define as:

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto y^2 - x(x-1)(x-a) \end{aligned}$$

5. It is not true that for a proper injective immersion $f : M \to N$, $\mathcal{O}$ open implies $f(\mathcal{O})$ is open in $N$, not even locally. And it is easy to find counterexamples, say $f : \mathbb{R} \to \mathbb{R}^2$ the inclusion as x-axes.
   In fact, we need to show that locally $\mathcal{O}$ open implies $f(\mathcal{O})$ is open in $f(M)$ with the topology induced from $N$. In other words, we need to show that there exists an open set $U$ in $N$ such that $f(\mathcal{O})$ is the intersection $f(M) \cap U$. Since locally near $\mathcal{O}$, by rank theorem, the map $f$ is a coordinate slice, so we can choose the open set $U$ as $U = N_\varepsilon(f(\mathcal{O}))$, a $\varepsilon$tubular neighborhood of $f(\mathcal{O})$, and then prove by contradiction that there exist an $\varepsilon$ such that $f(M) \cap N\varepsilon(f(\mathcal{O}) = f(\mathcal{O})$.

## Solutions

**Problem 1.** (a) A sequence $\{x_n\}$ in a topological space $X$ *converges to infinity* if, for each compact set $K \subset X$, there is an $N$ such that $x_n \notin K$ for all $n \geq N$.

(b) Suppose that $f : X \to Y$ is proper and $x_n \to \infty$ in $X$. Fix a compact set $K$ in $Y$. Since $f$ is proper, we know $f^{-1}(K)$ is compact. Hence there is an $N$ such that $x_n \notin f^{-1}(K)$ $\forall n \geq N$. But then $f(x_n) \notin K$ for all $n \geq N$. This means that $f(x_n) \to \infty$ in $Y$.

(c) One example is $f(x) = \sin x$, noting that $f^{-1}(0) = \{n\pi | n \in \mathbb{Z}\}$ is not compact. Another is $f(x) = \arctan x$, noting that $f^{-1}([0, \pi/2]) = [0, \infty)$ is not compact.

**Problem 2.** Let $S \subset \mathbb{R}^4$ be the surface defined by the equations $y = -x^2$ and $x^2+y^2+y+s^2+t^2 = 1$. Using the first equation to eliminate $y$, we can also write

$$S = \Big\{(x,y,s,t) \;\Big|\; y = -x^2 \text{ and } x^4+s^2+t^2 = 1\Big\}.$$

These equations define a function $f:\mathbb{R}^4 \to \mathbb{R}^2$ by $f(x,y,s,t) = \begin{pmatrix} x^2+y \\ x^4+s^2+t^2 \end{pmatrix}$. Then $S = f^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and

$$(Df)_p = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 4x^3 & 0 & 2s & 2t \end{pmatrix}$$

for $p = (x,y,s,t)$. The last three columns show that $Df$ has rank 1 whenever $s \neq 0$ or $t \neq 0$. In the remaining case $s = t = 0$, the condition that $p \in S$ implies that $x = 1$, and again $(Df)_p$ has rank 2. Thus $(Df)_p$ is surjective for each $p \in S$, so $S$ is a 2-dimensional submanifold of $\mathbb{R}^4$ by the Level Set Theorem.

To see that $S$ is diffeomorphic to $S^2$, let $\Sigma = \{(x,s,t) \in \mathbb{R}^3 \,|\, x^4+s^2+t^2 = 1\}$. Then

$$g:\Sigma \to \mathbb{R}^4 \quad \text{by } g(x,s,t) = (x,-x^2,s,t)$$

is a smooth injection whose image is $S$. Thus $S$ is diffeomorphic to $\Sigma$.

It remains to show that $\Sigma$ is diffeomorphic to $S^3$. Consider the map $G:\mathbb{R}^3 \to \mathbb{R}^3$ by

$$G(x,s,t) = (x',s',t') = \frac{1}{\sqrt{\phi(x)}}\Big(x,s,t\Big) \quad \text{where} \quad \phi(x) = 1+x^2(1-x^2).$$

The equation $x^4+s^2+t^2 = 1$ implies that $|x| \leq 1$, so $\phi(x) \geq 1$ at each $x \in \Sigma$, and hence $\phi^{-1/2}$ is a smooth function on $S$. This means that the restriction of $G$ to $\Sigma$ is a diffeomorphism from $\Sigma$ to $G(\Sigma)$. But

$$\begin{aligned}(x')^2+(s')^2+(t')^2 &= \phi^{-1}(x)\big(x^2+s^2+t^2\big) = \phi^{-1}(x)\Big((1+x^2-x^4)+x^4+s^2+t^2-1\Big) \\ &= 1 + \phi^{-1}(x)\big(x^4+s^2+t^2-1\big),\end{aligned}$$

so $G(x,y,z) = (x',s',t')$ lies on the unit sphere $S^3 \subset \mathbb{R}^3$ if and only if $(x,y,z) \in \Sigma$. Thus $G(\Sigma) = S^3$.

**Problem 3.** Consider the level set $F^{-1}(a)$ of solutions of $F(x,y) = x^3+xy+y^3 = a$. Then

$$(DF)_{(x,y)} = \big(3x^2+y,\; 3y^2+x\big)$$

has rank 1 unless $3x^2+y = 3y^2+x = 0$. Solving these equations simultaneously gives two solutions $p = (0,0)$ and $q = (-\frac{1}{3},-\frac{1}{3})$. Since $F(p) = 0$ and $F(q) = -\frac{1}{27}$, the Level Set Theorem implies that

- $F^{-1}(a)$ is an embedded 1-dimensional manifold (i.e. curve) for all $a \neq 0, -\frac{1}{27}$.
- $F^{-1}(0)$ is an embedded curve except at $p$, and $F^{-1}(-\frac{1}{27})$ is an embedded curve except at $q$.

To understand the geometry of $C = F^{-1}(0)$ near $p$, consider $C \cap B(0,\varepsilon)$ for small $\varepsilon$. Dilate by changing to coordinates $(z,w)$ defined by $(x,y) = (\varepsilon z, \varepsilon w)$. The equation $F(x,y) = 0$ becomes

$$0 = zw + \varepsilon(z^3+w^3)$$

As $\varepsilon \to 0$ this converges in the unit disk, to the set defined by $zw = 0$, which is the union of the two coordinate axes. Thus $C$ is NOT an embedded submanifold at $p$.

Similarly, for $C' = F^{-1}(-\frac{1}{27})$, consider $C' \cap B(q,\varepsilon)$. Dilate and translate by setting $x = \varepsilon z - \frac{1}{3}$, $y = \varepsilon w - \frac{1}{3}$. The equation $F(x,y) = -\frac{1}{27}$ becomes (after some algebra)

$$0 = -(z^2-zw+w^2) + \varepsilon(z^3+w^3).$$

Taking $\varepsilon \to 0$, this converges in the unit disk to the solution of $z^2 - zw + w^2 = 0$. This factors as $(z + \alpha w)((z + \alpha^{-1} w)$ where $\alpha^2 + \alpha + 1 = 0$, so the solution set is the union of two lines. Thus $C'$ is also NOT an embedded submanifold at $q$.

**Problem 4.** Given $f : M \to \mathbb{R}$, let $G$ be the graph $G = \{(x, f(x)) \in M \times \mathbb{R} \,|\, x \in M\}$. Define

$$F : M \times \mathbb{R} \to \mathbb{R} \quad \text{by} \quad F(x, y) = (y - f(x)).$$

Then $F$ is smooth and $DF = (Df, 1)$ has rank 1 at each point $(x, y)$. Hence $G$ is an embedded submanifold by the Level Set Theorem.

**Problem 5.** An embedding is an immersion that is a homeomorphism onto its image (giving the image the induced topology). The hypotheses imply that $f$ is continuous and that $f^{-1} : f(M) \to M$ is a bijection. It suffices to show $f^{-1}$ is continuous ($f$ is then a homeomorphism onto its image). But $g = f^{-1}$ is continuous $\Leftrightarrow$ $g^{-1}(A) = f(A)$ is closed for every closed set $A$ in $M$, i.e. if $f$ is a closed map. This is true by the following lemma, which also implies that the image $f(M)$ is closed.

**Lemma.** A proper continuous map $f : M \to N$ between manifolds is a closed map (i.e. images of closed sets are closed).

*Proof.* Given a closed set $A$ in $M$, we must show that $f(A)$ is closed. It suffices to show that if a sequence $y_n$ in $f(A)$ converges to a point $y_0 \in N$, then $y_0 \in f(A)$. The set $K = y_0 \cup \{y_n\}$ is compact (any sequence in $K$ has a convergent subsequence in $K$) so, since $f$ is proper, $f^{-1}(K)$ is compact. We can then choose an $x_n \in A \cap f^{-1}(K)$ with $f(x_n) = y_n$. Because $K$ is compact and $A$ is closed, there is a subsequence $x_{n_k}$ that converges to a point $x_0 \in A$. Since $f$ is continuous, $f(x_{n_k}) \to f(x_0) \in f(A)$. On the other hand, the subsequence $y_{n_k} = f(x_{n_k})$ converges of $y_0$. We conclude that $y_0 \in A$, as required. $\square$

**Problem 6.** First note that $M_a = \{(x, y) | y^2 = x(x-1)(x-a)\}$ is the level set $F^{-1}(0)$ of $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = y^2 - x(x-1)(x-a)$. Then

$$DF \;=\; (3x^2 - 2ax - 2x + a,\; 2y)$$

has rank 1 unless $3x^2 - 2ax - 2x + a = 0$ or $y = 0$. But the only points on $M_a$ with $y = 0$ have $x = 0, 1$ or $a$, and the first entry in $DF$ is non-zero at $(a, 0)$ only if $a = 0, 1$. Thus $(DF)_p$ is surjective except for the cases:

$$a = 0,\; p = (0, 0), \qquad \text{and} \quad a = 1,\; p = (1, 0).$$

By the Level Set Theorem, $M_a$ is an embedded curve for all $a \neq 0, 1$, and $M_0$ and $M_1$ are an embedded curves except possibly at $p = (0, 0) \in M_0$ and $q = (1, 0) \in M_1$.

- $M_0$ is defined by $y^2 = x^2(x-1)$. For $x$ near 0, $x - 1 \approx -1$, so the RHS is negative and hence there are no solutions except $(0, 0)$. Thus $p = (0, 0)$ is an isolated point of $M_0$. This means that there is no topology that makes $M_0$ into an immersed 1-dimensional submanifold.
- $M_1$ is defined by $y^2 = x(x-1)^2$. Translating coordinates by setting $z = x - 1$, this becomes $y^2 = z^2(z+1)$ with $q = (0, 0)$ is (z,y) coordinates. Now the RHS is positive, so $y = \pm\sqrt{z^2(z+1)} \approx \pm z$ for small $z$. Geometrically, this is two lines crossing at $q$. Hence $M_1$, with the induced topology, is an immersion, but not an immersed or embedded 1-dimensional submanifold.