Math 868 — Homework 3

Due Monday, Oct. 1

1. A map $f : M \to N$ is called *proper* if the inverse images of compact sets are compact (that is, K compact in $N \Rightarrow f^{-1}(K)$ is compact).

   (a) Invent a precise definition for the phrase "a sequence $\{x_k\}$ converges to infinity" in a topological space X.

   Your definition apply, in particular, to $X = \mathbb{R}^n$, should not mention distance functions, but instead should include the phrase "for every compact subset $K \subset X$".

   (b) Prove that if $f : X \to Y$ is proper and (with your definition) $x_k \to \infty$, then $f(x_k) \to \infty$.

   (c) Give an example of a smooth map $f : \mathbb{R} \to \mathbb{R}$ that is not proper.

2. Do Problem 5-1 on page 123 in Lee.

3. Do Problem 5-7 on page 123 in Lee.

4. Use the Preimage Theorem (called the "Regular Level Set Theorem" in Lee page 106) to prove that the graph of any smooth map $f : M \to \mathbb{R}$ is a closed embedded submanifold of $M \times \mathbb{R}$.

5. Prove that a proper one-to-one immersion $f : M \to N$ is an embedding with closed image. (This implies that if M is compact then all immersions are embeddings.)

   First observe that the hypotheses implies that $f^{-1} : f(M) \to M$ exists. It suffices to show f^{-1} is continuous. Then show

   (a) The hypotheses implies that for each $p \in M$ there is a local product chart around $f(p)$: $P : U \times B(\epsilon) \to N$ with $f(x) = P(x, 0)$. It suffices to show $\mathcal{O}$ open in U implies that $f(\mathcal{O})$ is open in N.

   (b) If this fails, there is a sequence $y_n = f(x_n)$ with $x_n \notin U$ but $y_n \to y_0 \in P(\mathcal{O} \times \{0\})$. This leds to a contradiction that the map is 1-1.

   A different approach (instead of doing (a) and (b)) is to use the fact that manifolds are locally compact (i.e. each point lies in a compact neighborhood) to prove that any proper map $f : M \to N$ between manifolds is *closed map*, i.e. images of closed sets are closed.

6. Do Problem 5-10 in Lee.