Math 868 — Homework 2

Due Friday, Sept. 21

1. Let $R = \{(x, y) \,|\, x > 0\}$ be the right half-plane and let $\Phi : \mathbb{R}^+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be the map $\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$ that changes into polar coordinates into (x, y) coordinates. Write down $D\Phi$ and $D\Phi^{-1}$ as matrices.

2. Use the matrices you found in Problem 1 above to express the vector $V = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}$ in polar coordinates, i.e. as a linear combination of $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial r}$.

3. In class we proved that T_pM is the set of velocity vectors at p for all paths in M through p. Use this to show:

   (a) Let $\phi : S^1 \to \mathbb{R}^2$ be the embedding of the unit circle and fix $p \in S^1$ with $\phi(p) = (a, b)$. Show that the image of $[D\phi]_p T_pS^1$ is the 1-dimensional subspace of R^2 spanned by $(-b, a)$.

   (b) Similarly, for the embedding $\phi : S^2 \to \mathbb{R}^3$ of the unit sphere and $p \in S^2$, find two vectors that are a basis of $[D\phi]_p T_pS^2$ at a point $\phi(p) = (a, b, c)$. (This can be done by simple geometry *but don't do that.* Find two paths and differentiate).

The next two problems use the following definitions and fact:

(a) A topological space X is *disconnected* if it is the disjoint union of two non-empty sets X_1 and X_2, each of which are both open and closed.

(b) X is *connected* if it is not disconnected.

(c) A manifold M is *path-connected* if for any two points $p, q \in X$ there is a smooth map $\sigma : [0, 1] \to M$ with $\sigma(0) = p$ and $\sigma(1) = q$.

One can show (rather easily) that there are exactly 10 types of connected subsets of $\mathbb{R}$: $\mathbb{R}$ itself, a single point $\{a\}$ and, for each $a < b$, the intervals $[a, b]$, (a, b), $(a, b]$, $[a, b)$, (a, ∞), $[a, \infty)$, $(-\infty, a)$, and $(-\infty, a]$.

4. Prove that a path-connected manifold is connected. *Use proof-by-contradiction.*

5. (Lee, Problem 3-1) Suppose that $f : M \to N$ is a smooth map between manifolds with M connected and that $Df_p : T_pM \to T_{f(p)}N$ is the zero map at each $p \in M$.

   (a) Prove that f is *locally constant*, i.e. each $p \in M$ has a neighborhood U such that $f(U)$ is a single point. *Use charts and the Fund. Theorem of Calculus.*

   (b) Prove that the image of f is a single point. *Again use proof-by-contradiction.*

6. If $f : M \to N$ is a diffeomorphism from an m-dimensional manifold to an n-dimensional manifold, prove that $m = n$. *Hint:* fix $p \in M$ and consider $(Df)_p$.

Brackets and Lie algebras (Lee, Chapter 8).

7. Suppose that vector fields X and Y are given in local coordinates $\{x^i\}$ by

$$X = \sum_i X^i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad Y = \sum_j Y^j \frac{\partial}{\partial x^j}.$$

   where the components X^i and Y^j are functions of the $\{x^i\}$. Write

$$[X, Y] = \sum Z^k \frac{\partial}{\partial x^k}.$$

   Find a formula for the components Z^k in terms of the X^i and Y^j. (This is in the textbook, but find it on your own).

8. Do parts (a) and (b) (skip (c)) of Problem 8-16 on page 201 in Lee.

9. (Lee, Problem 8-19) Show that $\mathbb{R}^3$ is a Lie algebra with $[X, Y] \overset{def}{=} X \times Y$.
   You may use the standard properties of cross product.

10. Do Problem 8-20 in Lee. Then sketch the vector field Z in the xy plane.