Math 868 — Homework 1

Due Friday, Sept. 7

For Problems 1 and 2, let (X, d) and (Y, d') be metric spaces. The questions refer to the following two versions of the definition of continuous map:

Definition 1. A map $f : X \to Y$ is *continuous* if, for each convergent sequence $x_n \to x_0$ in X, the corresponding seqence $f(x_n)$ converges to $f(x_0)$ in Y.

Definition 2. $f : X \to Y$ is continuous if and only if $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$ is open for every open set U in Y.

1. Fix a point x_0 in X. Using Definition 1, show that the function $f : X \to \mathbf{R}$ defined by $f(x) = d(x, x_0)$ is continuous.

2. Prove that Definition 1 is equivalent to Definition 2. Here is one way to do this:

   (a) First suppose that f is continuous in the sense of Definition 2, and that $x_n \to x_0$ is a convergent sequence in X. For each $\epsilon > 0$, the ball $B(f(x_0), \epsilon)$ in Y is open, so ....

   (b) Conversely, suppose that f is continuous in the sense of Definition 2. Fix an open set U in Y. Prove that $f^{-1}(U)$ is open by contradiction, as follows.

   If $\mathcal{O} = f^{-1}(U)$ is *not* open then there is a point $p \in \mathcal{O}$ such that no ball $B(p, \epsilon)$ is contained in $\mathcal{O}$. Hence for each $n = 1, 2, \ldots$, there is a point $x_n \in B(p, \frac{1}{n})$ that does not lie in $f^{-1}(U)$. Then $x_n \to p$ because ...

3. A subset $Z \subset X$ ic called *closed* if its complement $Z^c = \{x \in X | x \notin Z\}$ is open. Show that $f : X \to Y$ is continuous if and only if $f^{-1}(Z)$ is closed for every closed set Z in Y. (*Use (b) to show that each $x \notin f^{-1}(Z)$ lies in a ball that does not intersect $f^{-1}(Z)$*).

4. A *topological space* is a set X together with a collection $\mathcal{T}$ of subsets of X, called open sets, such that

   (a) X and the empty set $\emptyset$ are open.

   (b) The union of an arbitrary collection of open sets is open.

   (c) The intersection of finite collection of open sets is open.

   Let (X, d) is a metric space, and let $\mathcal{T}$ be the collection of open subsets of X as defined in class: $U \subset X$ is open if, for each $x \in U$, there is a $\delta > 0$ such that the ball $B(x, \delta)$ lies in U (this is called the topology "induced by the metric"). Prove that $(X, \mathcal{T})$ is a topological space.

   Hint: (a) holds by definition. For (b), show that $U_1 \cap U_2 \cap \cdots \cap U_n$ is open if each U_i is open, and for (c) show U_α open for all α in some index set A implies that $\bigcup_{\alpha \in A} U_\alpha$ is open.

5. A topological space $(X, \mathcal{T})$ is *Hausdorff* if for each pair of points $x, y \in X$ with $x \neq y$, there are open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

   Prove that a metric space, with the induced topology, is Hausdorff (this can be done in 2 lines).

Definition A function $f : \mathbf{R} \to \mathbf{R}$ is *smooth* or C^∞ if its derivatives $f^{(k)}(x)$ of all orders exist. Polynomials and $f(x) = e^x$ are smooth, and compositions of smooth functions are smooth.

6. This problem gives the steps for constructing a "C^∞ bump function". Pages 49-51 in Lee's book describe a similar — but not identical — construction.

   (a) An extremely useful function $f : \mathbf{R} \to \mathbf{R}$ is

   $$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

   Sketch the graph of f and prove that f is smooth at each $x \neq 0$ (*Note: $\phi(x)$ smooth $\Rightarrow e^{\phi(x)}$ smooth.*)

   (b) Read Lee's proof (pages 49-50 in the textbook) that f is also smooth at $x = 0$. *No need to write anything on this!*

   (c) Fix $0 < a < b$. Sketch the graph of $g(x) = f(x-a)f(b-x)$ and show that g is a smooth function, positive on the interval (a, b) and 0 elsewhere.

   (d) Sketch the graph of

   $$h(x) \;=\; \frac{\int_{-\infty}^{x} g \; dx}{\int_{-\infty}^{\infty} g \; dx}$$

   This is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$ and $0 < h(x) < 1$ for all $x \in (a, b)$ (no proof needed here).

   (e) Now construct a smooth "bump function" $\beta(x)$ on $\mathbf{R}^n$ that equals 1 on the ball $B(0, a)$, is zero outside the ball $B(0, b)$ and is strictly between 0 and 1 at the intermediate points.

Definition A map $f : \mathbf{R}^n \to \mathbf{R}^n$ is a *diffeomorphism* if it is 1-1, onto, smooth and f^{-1} is also smooth (equivalently, if f is a homeomorphism such that f and f^{-1} are smooth).

7. Prove that a smooth bijective map between manifolds need not be a diffeomorphism. In fact, show that following are examples.

   (a) $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3$.

   (b) $g : [0, 2\pi) \to S^1$ by $g(x) = e^{ix}$ (regarding S^1 as the unit circle in the complex plane). *Sketch, and show that ϕ^{-1} is defined but is not even continuous.*

8. Let S^n be the unit sphere in $\mathbf{R}^{n+1}$, with its north and south poles $n = (0, 0, \ldots, 1)$ and $s = (0, 0, \ldots, -1)$. Stereographic projection from the north pole is the map $\sigma_n : S^n \setminus \{n\} \to \mathbf{R}^n$ by

   $$\sigma_n(x^1, \ldots, x^{n+1}) \;=\; \frac{1}{1 - x^{n+1}} \Big(x^1, \ldots, x^n \Big).$$

   σ_s is given by the similar formula with $1 - x^{n+1}$ replaced by $1 + x^{n+1}$. It is straightforward to check that

   $$\sigma_n^{-1}(y^1, \ldots, y^n) \;=\; \frac{1}{1 + |y|^2} \Big(2y^1, \ldots, 2y^n, |y|^2 - 1 \Big).$$

   Show that $\{\sigma_n, \sigma_s\}$ is a atlas for a smooth structure on S^n, as follows:

   (a) What is the domain and range of $\sigma_s \circ \sigma_n^{-1}$?

   (b) Write down a formula for $\sigma_s \circ \sigma_n^{-1}$ and conclude (by inspection) that it is smooth.

   (c) Similarly write the formula for $\sigma_n \circ \sigma_s^{-1}$ and conclude that it is smooth.