SOLUTIONS

Dec 13, 2018

Math 868 — Final Exam

In this exam, all manifolds, maps, vector fields, etc. are smooth.

Part 1. Complete **5 of the following 7** sentences to make a precise definition (5 points each).

1. A map $f : X \to Y$ between manifolds is a *submersion* if

   $DF_p : T_pX \to T_{f(p)}Y$ is surjective for all $p \in X$.

2. A map $\pi : V \to M$ between manifolds is a *(locally trivial) vector bundle* of rank k if

   each $p \in M$ has a neighborhood U such that $\exists$ a diffeomorphism ϕ satisfying

   (i) $\pi_1 \circ \phi = \pi$, where $\pi_1 : U \times \mathbf{R}^k \to U$ is the projection onto the first factor, and

   (ii) The restriction of ϕ to each fiber $\pi^{-1}(x)$, $x \in U$, is linear.

3. Let X and Y be vector fields on M with flows $X \leftrightarrow \phi_t$ and $Y \leftrightarrow \psi_t$. The *Lie Derivative* of Y in the direction X is the vector field defined by

$$\mathcal{L}_X Y \;=\; \frac{d}{dt}(\phi_{-t})_* Y\Big|_{t=0} \qquad \text{or} \quad \lim_{t\to 0} \frac{(\phi_{-t})_* Y - Y}{t} \qquad \text{or} \quad \lim_{t\to 0} \frac{Y - (\phi_t)_* Y}{t}.$$

4. A *orientation form* for a manifold M is

   an n-form σ on M (where $n = \dim M$) that vanishes nowhere (i.e. $\sigma(x) \neq 0 \;\; \forall x \in M$).

5. Two maps $f, g : M \to N$ are *homotopic* if there is a continuous map

   $H : [0,1] \times M \to N$ such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ for all $x \in M$.

6. The *Poincaré Lemma* states that if a domain $\Omega \subset \mathbf{R}^n$ is

   contractible, then every closed p-form, $p > 0$, is exact (or, equivalently, then $H^p(\Omega) = 0 \; \forall p > 0$.)

7. Let (M, g) be an n-dimensional compact Riemannian manifold with volume form $dvol_g$.

   The Hodge star operator is the linear map $* : \Omega^p_M \to \Omega^{n-p}_M$ defined by $\omega \mapsto *\omega$, where $*\omega$ is the unique element of Ω^{n-p}_M such that

$$\eta \wedge *\omega \;=\; \langle \eta, \omega \rangle \; dvol_g \qquad \forall \eta \in \Omega^p_M.$$

Part 2. Do **all 4** of the following Short Problems (8 points each).

1. Give a precise definition of a *smooth manifold*.

   A smooth manifold M of dimension n is a metrizable (or second countable, Hausdorff) space with a maximal collection of maps ("charts") $\{\phi_\alpha : U_\alpha \to V_\alpha \,\big|\, \alpha \in A\}$ indexed by a set A that satisfy:

   (a) $U_\alpha \subset M$ and $V_\alpha \subset \mathbf{R}^n$ are open,

   (b) $\{U_\alpha\}$ is a cover of M.

   (c) Each ϕ_α is a homeomorphism.

   (d) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a diffeomorphism.

   Caution: In (c), one can't say ϕ_α is a diffeomorphism because at that point, U_α is only a topological space.

2. Consider $\mathbf{R}^4$ with coordinates (w, x, y, z). Let $f : \mathbf{R}^2 \to \mathbf{R}^4$ be defined by $f(u, v) = (u^2 v, v, u, v^3)$. Define a 2-form on $\mathbf{R}^4$ by

$$\alpha = w^2\ dy \wedge dz + y^3\, dy \wedge dw - 2yw\ dz \wedge dw.$$

   (a) Compute $f^*\alpha$. Noting that $dw = d(u^2 v) = 2uvdu + u^2 dv$, $dx = dv$, $dy = du$, and $dz = d(v^3) = 3v^2 dv$, and that $du \wedge du = 0$ and $dv \wedge dv = 0$, one calculates

$$\begin{aligned} f^*\alpha &= (u^2 v)^2\, du \wedge 3v^2\, dv \ +\ u^3\, du \wedge (\cancel{2uv\,du} + u^2\, dv) - (2u^3 v)(3v^2\, dv) \wedge (2uvdu + \cancel{u^2 dv}) \\ &= (3u^4 v^4 + u^3)\ du \wedge dv \ -\ 12u^4 v^4\ dv \wedge du \\ &= (15u^4 v^4 + u^5)\ du \wedge dv. \end{aligned}$$

   (b) Find a 1-form β on $\mathbf{R}^4$ such that $\alpha = d\beta$. There are many possible answers, including

$$\beta = w^2 y\, dz + \frac{y^4}{4}\, dw \qquad \text{and} \qquad \beta = w^2 y\, dz - y^3 w\, dy.$$

3. (a) Suppose that $f : M \to N$ is a diffeomorphism between manifolds. Prove that at each point p, Df_p is an isomorphism of the tangent spaces.

   By assumption, there is a smooth inverse map $g : N \to M$ with $f \circ g = Id.$ and $g \circ f = Id.$ By the Composite Function Theorem,

$$D(f \circ g) = Df \circ Dg = D(Id) = Id \qquad \text{and similarly} \qquad D(g \circ f) = Id.$$

   Hence $Df_p : T_p M \to T_{f(p)} N$ is a (linear) isomorphism of vector spaces.

   (b) Prove that $\mathbf{R}^k$ is not diffeomorphic to $\mathbf{R}^n$ if $k \neq n$.

   Fix $p \in \mathbf{R}^k$. If $\phi : \mathbf{R}^k \to \mathbf{R}^n$ is a diffeomorphism, by by (a) $D\phi_p : T_p\mathbf{R}^k \to T_{f(p)}\mathbf{R}^n$ is a vector space isomorphism. Hence $\dim T_p\mathbf{R}^k = k$ is equal to $\dim T_{f(p)}\mathbf{R}^n = n$.

4. (a) Define the DeRham cohomology group $H^p(M)$.

$$H^p(M) = \frac{\ker\ d: \Omega^p_M \to \Omega^{p+1}_M}{im\ d: \Omega^{p-1}_M \to \Omega^p_M} = \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}}$$

(b) Prove that a map $f: M \to N$ between manifolds induces a map $f^*: H^p(N) \to H^p(M)$ for each p.

Let $A^p(M) \subset \Omega^p_M$ denote the vector subspace of closed p-froms. For each $\omega \in A^p(N)$, the pullback $f^*\omega$ is closed (since $d(f^*\omega) = f^*d\omega = 0$), so defined a class $[f^*\omega]$ in $H^p(M)$. Define a map

$$L: A^p(N) \to H^p(M)$$

by $L(\omega) = [f^*\omega]$. This map is linear because

$$L(a\omega + b\omega') = [f^*(a\omega + b\omega')] = [af^*\omega + bf^*\omega'] = a[f^*\omega] + b[f^*\omega'].$$

It is also constant on equivalence classes in $A^p(N)$ because

$$L(\omega + d\eta) = [f^*(\omega + d\eta)] = [f^*\omega] + [f^*d\eta] = [f^*\omega] + [df^*\eta] = [f^*\omega].$$

Hence L induces a linear map $H^p(N) \to H^p(M)$.

Part 3. Do **4 of the remaining 7** longer problems (11 points each).

5. Use the Preimage Theorem (called the "Regular Level Set Theorem"' in Lee) to prove that the graph of any smooth map $f: M \to \mathbf{R}$ between manifolds is a closed embedded submanifold of $M \times \mathbf{R}$.

Define $F: M \times \mathbf{R} \to \mathbf{R}$ by $F(x,t) = f(x) - t$. Then $f^{-1}(0)$ is the graph $G_f = \{(x,t) \in M \times \mathbf{R} \,\big|\, t = f(x)\}$, F is smooth (it is the sum of two smooth functions), and DF_p is surjective at each $p = (x,t) \in G_f$ because $DF_p(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t}$. Hence G_f is a closed embedded submanifold by the Regular Value Theorem.

6. Let S be the 2-manifold with boundary consisting of the points of the "monkey saddle" graph $\{(x,y,z) \,|\, y^3 - 3x^2y - z = 0\}$ whose (x,y) coordinates lie in the ellipsoidal region

$$E = \left\{(x,y) \in \mathbf{R}^2 \,|\, x^2 + \left(\frac{y}{2}\right)^2 \leq 1\right\}.$$

(a) Write down a diffeomorphism $f: D \to S$, where D is the unit disk in $\mathbf{R}^2$ with coordinates (u,v).

Set $f(u,v) = (u, 2v, 8v^3 - 6u^2v)$. This is surjective because S is the graph $z = y^3 - 3x^2y$ over E, and is injective with inverse $f^{-1}(x,y,z) = (x, y/2)$. Thus f is a diffeomeorphism (f and f^{-1} are polynomials, so are smooth).

(b) Orient S with the orientation form $\sigma = dx \wedge dy$, and D with the orientation form $du \wedge dv$. Is your map f positively oriented? $f^*\sigma = du \wedge d(2v) = 2du \wedge dv$. Since $2 > 0$, f is positively oriented.

(c) Compute $\displaystyle\int_S x\ dx \wedge dy + y^2\ dx \wedge dz$ = $\int_S xdx \wedge dy + y^2 dx \wedge dz = \int_D u(du \wedge 2dv) + (2v)^2(du \wedge d(8v^3 - 6u^2v))$. Noting that $d(8v^3 - 6u^2v)) = 24v^2dv - 12uvdu - 6u^2dv$ and $du \wedge du = 0$, this reduces to $\int_D 2(u+48v^4-12u^2v^2)\,du\wedge dv$. Note that $\int_D u=0$ because u is an odd function. Now switch to polar coordinates by $u=r\cos\theta, v=r\sin\theta$ and $du\wedge dv = rdrd\theta$ (since f is positively oriented):

$$= 2\cdot 12\int_0^{2\pi}\int_0^1 r^4\left(4\sin^4\theta-\sin^2\theta\cos^2\theta\right)\,rdrd\theta \;=\; \frac{24}{6}\int_0^{2\pi} 4\sin^4\theta-\sin^2\theta\cos^2\theta\;d\theta$$

Integrating by parts using $(\sin^3\theta\cos\theta)'=-\sin^4\theta+3\sin^2\theta\cos^2\theta$, this becomes

$$4\int_0^{2\pi} 11\sin^2\theta\cos^2\theta\;d\theta \;=\; \int_0^{2\pi} 11(2\sin\theta\cos\theta)^2\;d\theta \;=\; 11\int_0^{2\pi}\sin^2 2\theta\;d\theta \;=\; 11\pi.$$

7. This problem is about the definition of vector fields *as derivations.*

   (a) Complete the definition: A *vector field* is a linear map $X: C^\infty(M)\to C^\infty(M)$ such that $X(fg)=Xf\cdot g+f\cdot Xg \quad \text{for all } f,g\in C^\infty(M).$

   Now let $\{x^i\}$ be local coordinates on an open set $U\subset M$.

   (b) Use your answer to (a) to show that $[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}]=0$ for all i,j. $\frac{\partial}{\partial x^i}$ is the vector field defined by $\frac{\partial}{\partial x^i}(f)=\frac{\partial f}{\partial x^i}$. Then $[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}]f=\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(f)-\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}(f)=\frac{\partial f}{\partial x^i\partial x^j}-\frac{\partial f}{\partial x^j\partial x^i}=0$.

   (c) Show that if $X=\sum X^i\frac{\partial}{\partial x^i}$ and $Y=\sum Y^i\frac{\partial}{\partial x^i}$ then $\quad [X,Y]=\sum\left(X^i\frac{\partial Y^j}{\partial x^i}-Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$.

   Write $\frac{\partial}{\partial x^i}$, as ∂_i. Then $\sum_{i,j}[X^i\partial_i, Y^j\partial_j]f$ is $X^i\partial_i Y^j\partial_j f + X^iY^j\partial_i\partial_j f - Y^j\partial_j X^i\partial_i f - Y^jX^i\partial_j\partial_i f$. Omitting the f, changing indices on the thrid term and using (a), this becomes

$$(X^i\partial_iY^j\partial_j - Y^j\partial_iX^i)\partial_j + X^iY^j[\partial_i,\partial_j] \;=\; (X^i\partial_iY^j-\partial_jY^j\partial_iX^i)\partial_j.$$

8. Let ω be an n-form on a compact n-dimensional manifold M with orientation form σ.

   (a) Write down a precise definition of the integral $\displaystyle\int_M\omega$.

   For an n-form η compactly supported on an open set $U\subset\mathbf{R}^n$, write $\eta=f\,dx^1\wedge\cdots\wedge dx^n$ for some function f, and define $\int_U\omega$ to be the ordinary integral $\int f$. This is well-defined up to sign by the change-of-variables formula from calculus.

   In general, choose smooth diffeomorphisms $\phi_\alpha: U_\alpha\to V_\alpha$ *into* M, with $U_\alpha\subset\mathbf{R}^n$ and $\{V_\alpha\}$ covering M. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{V_\alpha\}$. Define

$$\int_M\omega \text{ to be } \sum_\alpha\int_{U_\alpha}\pm\phi_\alpha^*(\rho_\alpha\omega)$$

   where, for each α, the sign $\pm$ are determined by orientation of ϕ_α.

   (b) Describe how to determine the sign.

   For each α, $\phi_\alpha^*\sigma$ is an n-form on a domain in $\mathbf{R}^n$, so can be written $\phi_\alpha^*=g\,dx^1\wedge\cdots\wedge dx^n$, and g vanishes nowhere on U_α because ϕ_α is a diffeomorphism. Then the sign is $+$ if $g>0$, and is $-$ is $g<0$ on U_α.

9. (a) What is the DeRham cohomology of the circle S^1? (do not prove).

$$H^p(S^1) = \begin{cases} \mathbf{R} & p = 0 \\ \mathbf{R} & p = 1 \\ 0 & p \neq 0, 1 \end{cases}$$

Use this and the Mayer-Vietoris sequence to find $H^*(S^2)$, as follows:

(b) Draw a picture showing your choice of U and V. Take U to be the northern 3/4 of S^2 and V to be the southern 3/4 (or $U = S^2 \setminus \{\text{south pole}\}$ and $V = S^2 \setminus \{\text{north pole}\}$).

(d) Write down the Mayer-Vietoris sequence relating $H^*(U), H^*(V)$ and $H^*(U \cap V)$.

$$\cdots \to H^p(S^2) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \to H^{p+1}(S^2) \to \cdots$$

(c) From (a) and the axioms of cohomology, what are $H^*(U), H^*(V)$ and $H^*(U \cap V)$?

Note that U and V are contractible, and that $U \cap V$ retracts to the equator. Hence by the Homotopy axiom and the Point axiom

$$H^p(U) = H^p(V) = H^p(\text{point}) = \begin{cases} \mathbf{R} & p = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and $H^*(U \cap V) = H^*(S^1)$ is an in (a).

(d) Use the resulting long exact sequence to find $H^*(S^2)$.

S^2 is connected, so $H^0(S^2) = \mathbf{R}$. The Mayer-Vietoris sequence above is, in part,

$$0 \to H^0(S^2) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(S^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to H^2(S^2) \to 0.$$

Using (c) this reduces to

$$0 \to \mathbf{R} \to \mathbf{R} \oplus \mathbf{R} \to \mathbf{R} \to H^1(S^2) \to 0 \oplus 0 \to \mathbf{R} \to H^2(S^2) \to 0.$$

In particular,

(i) $0 \to \mathbf{R} \to H^2(S^2) \to 0$ is exact $\implies H^12(S^2) = \mathbf{R}$.

(ii) For $p > 2$, $0 \to H^p(S^2) \to 0 \implies H^p(S^2) = 0$.

(iii) Counting dimensions in the exact sequence $0 \to \mathbf{R} \to \mathbf{R} \oplus \mathbf{R} \to \mathbf{R} \to H^1(S^2) \to 0$ shows that $H^1(S^2) = 0$.

Thus

$$H^p(S^2) = \begin{cases} \mathbf{R} & p = 0 \\ \mathbf{R} & p = 2 \\ 0 & p \neq 0, 2 \end{cases}$$

10. (a) Complete the definition: M be a smooth n-manifold with boundary if... See the definition in Lee.

(b) Now suppose that M is a smooth n-manifold with boundary. Show that ∂M is a smooth $(n-1)$-manifold without boundary and that the inclusion $\partial M \to M$ is a smooth embedding. See the definition in Lee.

11. The exterior derivative d is a linear map $d : \Omega^p_M \to \Omega^{p+1}_M$ for each $p \geq 0$ such that

(i) $d^2 = \underline{\ 0\ }$

(ii) For $f \in C^\infty(M)$, the 1-form df is defined by $df(X) = \underline{\ Xf\ }$ for all vector fields X.

(iii) $d(\omega \wedge \eta) = \underline{d\omega \wedge \eta + (-1)^p \omega \wedge d\eta}$ $\quad \forall \omega \in \Omega^p_M, \eta \in \Omega^q_M$.

Prove that these properties determine d uniquely, as follows:

(a) Fill in the blanks above.

(b) Show that these properties determine $d\omega$ for a 1-form ω. *Hint: write ω in local coordinates.*

First suppose that ω has support in one coordinate chart $\{x^i\}$. For each i, define a function ω_i by $\omega_i = \omega(\frac{\partial}{\partial x^i})$. Then $\omega = \sum w_i\, dx^i$ as follows: for any vector field $X = \sum X^j \frac{\partial}{\partial x^j}$,

$$\omega(X) = \sum x^j \omega(\tfrac{\partial}{\partial x^j}) = \sum X^j \omega_j, \text{while } \left(\sum w_i\, dx^i\right)(X) = \sum\sum \omega_i X^j dx^i(\tfrac{\partial}{\partial x^j}) = \sum \omega_j X^j.$$

Then by (i)–(iii), $d\omega = \sum d\omega_i \wedge dx^i - \omega_i ddx^i = \sum d\omega_i \wedge dx^i$; in particular, $d\omega$ is determined by (i)–(iii).

In general, let $\{U_\alpha\}$ be a coordinate charts that cover M with subordinate partition of unity $\{\rho_\alpha\}$. Then $\sum_\alpha \rho_\alpha = 1$, and hence $\omega = \sum \omega_\alpha$ where $\omega_\alpha = \rho_\alpha \omega$ is a 1-form supported on a coordinate chart. Then $d\omega = \sum d\omega_\alpha$ (by linearity), and each $d\omega_\alpha$ is determined by (i)–(iii) as above.

(c) Use induction to prove that these properties determine $d\omega$ for any p-form ω.

This is true for $p = 0, 1$ by parts (a) and (b). Assume inductively that it is true for $p-1$. Fix a p-form ω. As in (b), we can use a partition of unity to show that it suffices to assume that ω has support in a chart with coordinates x^i. For each i, let η_i be the $(p-1)$ form

$$\eta_i \ = \iota_{\frac{\partial}{\partial x^i}} \omega$$

Then $\omega = \sum_i \eta_i \wedge dx^i$. Hence by (iii) and (i), $d\omega = \sum d\eta_i \wedge dx^i + 0$, so $d\omega$ is well-defined and unique by the induction hypothesis.